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Singular Perturbations of First-Order Hyperbolic Systems with Stiff Source Terms

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This work develops a singular perturbation theory for initial-value problems of nonlinear first-order hyperbolic systems with stiff source terms in several space

2×2-systems and the well-known time-like condition for one-dimensional scalar second-order hyperbolic equations with a small positive parameter multiplying the highest derivatives. Under this stability condition, we construct formal asymptotic approximations of the initial-layer solution to the nonlinear problem. Furthermore, assuming some regularity of the solutions to the *limiting inner problem* and the *reduced problem*, we prove the existence of classical solutions in the uniform time interval where the reduced problem has a smooth solution and justify the validity of the formal approximations in any fixed compact subset of the uniform time interval. The stability condition seems to be a key to problems of this kind and can be easily verified. Moreover, this presentation unifies and improves earlier works for some specific equations. © 1999 Academic Press

Key Words: singular perturbations; first-order hyperbolic systems; structural stability condition; zero relaxation limit.

1. INTRODUCTION

This work deals with initial-value problems of nonlinear first-order hyperbolic systems with stiff source terms in several space variables:

$$U_t+\sum_{j=1}^{d} A_j(U)\, U_{x_j}=\frac{Q(U)}{\varepsilon},$$
$$U(x,0)=\bar{U}(x,\varepsilon). \tag{1.1}$$

Here U is the unknown n-vector function of $(x,t)\equiv(x_1, x_2, \cdots, x_d, t)\in \mathbf{R}^d\times[0,+\infty)$, $A_j=A_j(U)(j=1,2,\ldots,d)$ and $Q=Q(U)$ are the respective $n\times n$-matrix and n-vector smooth functions of $U\in G\subset\mathbf{R}^n$ (an open set

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called *state space*), $\bar{U}(x, \varepsilon)$ is a given initial value function, and ε is a small positive parameter. The subscripts (except j) denote the corresponding partial derivatives. For simplicity, we assume that $A_j (j=1, 2, \cdots, d)$ and Q do not depend on x, t, and ε; moreover, $\bar{U}(x, \varepsilon)$ is periodic in x with period $(1, 1, ..., 1) \in \mathbf{R}^d$. For the general cases, the reader may consult my thesis [22].

The aim of this work is to investigate the limiting problem as ε goes to zero from the viewpoint of singular perturbations. The basic assumption is that the source term $Q(U)$ has a nonempty *equilibrium manifold*

$$\mathscr{E} := \{U \in G: Q(U) = 0\}.$$

Because initial-value problems are under consideration, initial-layer phenomena are one of our main concerns.

First-order hyperbolic systems with stiff source terms model a large number of different physical phenomena. In particular, important examples occur in the kinetic theory [18], inviscid reactive flow [19, 13], magnetohydrodynamics, inviscid gas dynamics with relaxation, traffic flow, river flow, certain chemical exchange processes, and so on. For the others we refer to Chapters 3 and 10 of [21].

Next, we present our main results together with some historical comments. One of the main contributions of this work is the following observation that under reasonable assumptions, many equations of classical physics of the form (1.1) admit the following structure (also called *stability condition*):

(i) There is an invertible $n \times n$ matrix $P(U)$ and an invertible $r \times r (0 < r \leqslant n)$ matrix $\hat{S}(U)$, defined on the equilibrium manifold $\mathscr{E}$, such that

$$P(U) \, Q_U(U) = \begin{pmatrix} 0 & 0 \\ 0 & \hat{S}(U) \end{pmatrix} P(U) \qquad \text{for} \quad U \in \mathscr{E};$$

(ii) as a system of first-order partial differential equations, (1.1) is symmetrizable hyperbolic, that is, there is a positive definite Hermitian matrix $A_0(U)$ such that

$$A_0(U) \, A_j(U) = A_j^*(U) \, A_0(U) \qquad \text{for} \quad U \in G \quad \text{and} \quad j = 1, 2, ..., d;$$

(iii) the hyperbolic part and the source term are coupled in the sense

$$A_0(U) \, Q_U(U) + Q_U^*(U) \, A_0(U) \leqslant -P^*(U) \begin{pmatrix} 0 & 0 \\ 0 & I_r \end{pmatrix} P(U) \qquad \text{for} \quad U \in \mathscr{E}.$$

Here Q_U is the Jacobian of Q, the superscript "*" denotes the transpose operator acting on matrices and I_r is the r-order unit matrix. We know from [6] that (i) is just the usual assumption in the corresponding theory for ordinary differential equations, that is, $d=0$.

At this point, several comments are proper. The above stability condition is equivalent to the well-known subcharacteristic condition in [21, 14] for one-dimensional 2×2-systems with $r=1$ and the well-known time-like condition used in [8, 5, 7] for one-dimensional scalar second-order hyperbolic equations with a small positive parameter multiplying the highest derivatives. See also [20]. In addition, it has been pointed out respectively in [19, 1] that a simplified model for reacting flow and the one-dimensional Broadwell model of the Boltzmann equation, both consisting of three equations with $r=1$, satisfy a *weaker stability condition*, which consists of (i), (ii) and

(iii)' the hyperbolic part and the source term are coupled in the sense

$$A_0(U)\,Q_U(U) + Q_U^*(U)\,A_0(U) \leqslant 0 \qquad \text{for} \quad U \in \mathscr{E}.$$

More stability conditions for the system of the form (1.1) can be found in my thesis [22] and they provide the reader how the above stability condition was proposed.

After the completion of this work in 1992 the author got to know [2] (and [3]) where a notion of a strictly convex entropy is introduced for the system of the form (1.1) with the source term admitting a nontrivial constant *annihilator*. We note that there exist physical examples in magnetohydrodynamics whose source terms have not such an annihilator. On the other hand, the following has been pointed out in the proof of Theorem 2.2 in [2] that the existence of a convex entropy implies the above weaker stability condition. We will prove that the weaker one implies the stability condition if $r=1$ or $A_0(U)\,Q_U(U)$ is Hermitian.

We also remark that Kreiss in [12] studied the system of the form (1.1) under the assumption that A_j ($j=1, 2, ..., d$) are Hermitian and Q_U is skew Hermitian. The latter implies that all eigenvalues of Q_U are pure imaginary, while in present case Q_U has always eigenvalues with non-zero real parts ($r>0$).

In addition, the author mentions that this work was completed in 1992, see [22]. Afterwards, many papers on this kind of problems have shown up (see, e.g., [9, 16]). In particular, it is worthwhile to point out that based on the subcharacteristic condition, the authors in [9] proposed a new approach to construct shock-capturing numerical schemes, without using Riemann solvers, for conservation laws. It seems that the above stabilty condition plays a key role in studying the problems of this kind.

See [23] to know that the stability condition implies the stability of relaxation shock profiles for the simplified model system for reacting flow in [19] and [13].

Suppose the structural stability condition holds; the *limiting inner problem* (x is a parameter here)

$$\frac{d\tilde{I}}{d\tau} = Q(\tilde{U}) \qquad \text{with} \quad \tilde{I}(x, 0) = \bar{U}(x, 0) \tag{1.2}$$

has a unique solution $\tilde{I}(x, \tau)$ defined on $[0, +\infty)$ whose limit $\tilde{I}(x, +\infty)$ exists; and the equilibrium manifold can be expressed as

$$\mathscr{E} = \{ U = E(u) : u \in \mathscr{D} \subset \mathbf{R}^{n-r} \}, \tag{1.3}$$

where $\mathscr{D}$ is open and E is a smooth mapping from $\mathscr{D}$ to $\mathbf{R}^n$ such that the $n \times (n-r)$ matrix $E_u(u)$ is of full-rank. We use the matching expansion method in [17] to construct formal asymptotic approximations, of the initial-layer solution U^ε to the problem in (1.1), of the form

$$U_\varepsilon^m = \sum_{k=0}^{m} \varepsilon^k U_k(x, t) + \sum_{k=0}^{m} \varepsilon^k I_k(x, t/\varepsilon) \tag{1.4}$$

with $m \geqslant 1$ appropriately given. By our construction based on the *matching principle*, the leading term $I_0(x, t/\varepsilon)$ of the *initial-layer correction* (the second sum) is

$$I_0(x, t/\varepsilon) = \tilde{I}(x, t/\varepsilon) - \tilde{I}(x, +\infty) \tag{1.5}$$

and the leading term U_0 of the *outer expansion* (the first sum) satisfies the initial condition

$$U_0(x, 0) = \tilde{I}(x, +\infty).$$

Moreover, U_0 solves the *reduced problem*

$$\begin{aligned} Q(U_0) &= 0, \\ P^I(U_0)\left(U_{0t} + \sum_{j=1}^{d} A_j(U_0)\, U_{0x_j}\right) &= 0, \\ U_0(x, 0) &= \tilde{I}(x, +\infty). \end{aligned} \tag{1.6}$$

Here and below, $P^I(U_0) = P^I$ (resp. P^{II}) denotes the $(n-r) \times n$ (resp. $r \times n$) matrix consisting of the first $(n-r)$ (resp. last r) rows of $P(U_0)$.

The system of equations in (1.6) is called the *equilibrium system*. In order to solve this equilibrium system, one might be tempted to differentiate the first equation in (1.6), use (i) of the stability condition to get $P^{II}(U_0)\, U_{0t} = 0$, and then combine the second line in (1.6) to obtain

$$P(U_0)\, U_{0t} + \begin{pmatrix} P^I(U_0) \\ 0 \end{pmatrix} \sum_j A_j(U_0)\, U_{0x_j} = 0,$$

$$U_0(x, 0) = \tilde{I}(x, +\infty).$$

However, this method has some shortcomings, while it works for ordinary differential equations in [6]. First of all, $Q(U_0) = 0$ could not be deduced from the last equations. Recall that (i) of the stability condition is only valid on the equilibrium manifold. Thus, the equations are not well-defined, since P is only defined on the manifold. Moreover, the above initial-value problem may not be well-posed, since the system of equations is not necessarily hyperbolic. The last statement can be seen by considering the following trivial example

$$u_t + v_x = 0,$$

$$v_t + u_x = -v/\varepsilon.$$

Thus, we simply make the assumption in (1.3), which is convenient and for our latter examples also appropriate.

Assume some regularity of the solutions to the limiting inner problem and the reduced problem. We use energy methods based on the stability conditions to show the validity of formal asymptotic approximations (1.4) in any finite time interval where problem (1.1) has a smooth solution. On the basis of this validity result, we prove the existence of classical solutions U^ε in the ε-independent time interval where the reduced problem has a smooth solution. Here the main tool is the existence theory of classical solutions (local in time) for symmetrizable hyperbolic systems in [15].

A direct corollary of the above validity result is

$$U^\varepsilon = \sum_{k=0}^{m} \varepsilon^k U_k(x, t) + \sum_{k=0}^{m} \varepsilon^k I_k(x, t/\varepsilon) + O(\varepsilon^{m+1/2}) \tag{1.7}$$

for $x \in \mathbf{R}^d$ and t bounded. In particular, it follows from (1.7) and (1.4) that, out of the initial-layer, the solution U^ε of the original problem (1.1) converges to U_0, that of the reduced problem (1.6), as ε goes to zero. Note that the error estimate improves that of [1] for the one-dimensional Broadwell model of the Boltzmann equation, in that the convergence rate in (1.7) is independent of the dimension d and we require only $m \geqslant 1$ for any $d \geqslant 0$.

Since our approach is based on the assumed existence of smooth solutions to the equilibrium systems, we cannot show the validity of the formal asymptotic approximations in (1.4) when shock discontinuities are present. For non-smooth solutions, the convergence of U^ε to U_0 is still a topic of ongoing research efforts and, to my knowledge, the existing results are valid only for some special systems of the form (1.1). See [16] and references cited therein.

This paper is organized as follows. Section 2 is concerned with some relations between our stability condition and the weaker one, the strictly convex entropy condition in [2], and the subcharacteristic condition in [21, 14]. In Section 3, we show that the stability condition is satisfied by some important physical examples. Formal asymptotic approximations are constructed in Section 4 and estimated in Section 5. In Section 6 we justify the validity of the asymptotic approximations and prove the existence result.

2. ON THE STABILITY CONDITION

In this section we point out some relations between our stability condition and the weaker one, the strictly convex entropy condition in [2], and the subcharacteristic condition in [21, 14].

We begin with the following elementary fact.

LEMMA 2.1. *Let A_{12} and A_{22} be $(n-r)\times r$-, $r\times r$-matrices, respectively. Assume*

$$\begin{pmatrix} 0 & A_{12} \\ A_{12}^* & A_{22} \end{pmatrix} \leqslant 0.$$

Then $A_{12}=0$.

Proof. Assume $A_{12}\neq 0$. Then there exists an r-vector ξ_2 so that $A_{12}\xi_2 \neq 0$. Moreover, there exists an $(n-r)$-vector ξ_1 so that $\operatorname{Re}\xi_1^* A_{12}\xi_2\neq 0$. Let μ be a real number and take

$$\xi=\begin{pmatrix} \mu\xi_1 \\ \xi_2 \end{pmatrix}.$$

By the assumption we have

$$\xi^*\begin{pmatrix} 0 & A_{12} \\ A_{12}^* & A_{22} \end{pmatrix}\xi \leqslant 0,$$

that is, $2\mu \operatorname{Re}\xi_1^* A_{12}\xi_2+\xi_2^* A_{22}\xi_2\leqslant 0$ for all real μ. This is obviously false. ∎

THEOREM 2.2. *Assume the system in* (1.1) *satisfies the weaker stability condition, that is,* (i), (ii), *and* (iii)' *in the previous section. Then* $P^{-*}(U)\, A_0(U)\, P^{-1}(U)$ *is a block-diagonal matrix* (*with the same partition as that in* (i) *and* (iii)) *and there is a positive semidefinite Hermitian* $r\times r$ *matrix* $\Lambda(U)$ *such that*

$$A_0(U)\, Q_U(U)+Q_U^*(U)\, A_0(U)=-P^*(U)\begin{pmatrix}0 & 0\\ 0 & \Lambda(U)\end{pmatrix}P(U).$$

Furthermore, if either $r=1$ *or* $A_0(U)\, Q_U(U)$ *is Hermitian, then* $\Lambda(U)$ *is positive definite.*

Note that (i) always holds if the rank of $Q_U(U)$ for $U\in\mathscr{E}$ is equal to the number of its non-zero eigenvalues. This theorem provides a method to check whether a given system satisfies the stability condition.

Proof. Set $A=P^{-*}(U)\, A_0(U)\, P^{-1}(U)$. Since PQ_UP^{-1} is block-diagonal due to (i), the inequality in (iii)' implies that

$$\begin{aligned}0&\geqslant P^{-*}(U)[A_0(U)\, Q_U(U)+Q_U^*(U)\, A_0(U)]\, P^{-1}(U)\\ &=\begin{pmatrix}0 & A_{12}\hat{S}\\ \hat{S}^*A_{12}^* & A_{22}\hat{S}+\hat{S}^*A_{22}\end{pmatrix}.\end{aligned}\tag{2.1}$$

By Lemma 2.1 we see $A_{12}\hat{S}=0$. Since $\hat{S}$ is invertible, $A_{12}=0$ and therefore A is block-diagonal. Moreover, (2.1) shows that $A_{22}\hat{S}+\hat{S}^*A_{22}$ is negative semidefinite. Taking $\Lambda=-A_{22}\hat{S}-\hat{S}^*A_{22}$, we prove the first part.

For the other part, it suffices to show that Λ is invertible. In fact, if $A_0(U)\, Q_U(U)$ is Hermitian, then so is $A_{22}\hat{S}$. Therefore $A_{22}\hat{S}+\hat{S}^*A_{22}=2A_{22}\hat{S}$ is invertible, since A_{22} is positive definite and $\hat{S}$ is invertible. In addition, it is obvious that $A_{22}\hat{S}+\hat{S}^*A_{22}$ is invertible in case $r=1$. ∎

Remark 2.1. Indeed, there exist models for which $A_{22}\hat{S}+\hat{S}^*A_{22}$ is not invertible. To see this, let us consider the system in (1.1) with

$$d=2,\qquad n=2,\quad A_1=\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix},\quad A_2=\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix},\quad\text{and}\quad S=\begin{pmatrix}0 & \sqrt{2}\\ -\sqrt{2} & -2\end{pmatrix}.$$

Since A_1 is diagonal with distinct diagonal entries and A_2 is symmetric, the symmetrizer A_0 must be a scalar matrix. Without loss of generality, we take A_0 to be the unit matrix. On the other hand, two eigenvalues of S are $-1+i$ and $-1-i$, thus S is stable and $r=2$. However,

$$A_0S+S^*A_0=\begin{pmatrix}0 & 0\\ 0 & -4\end{pmatrix}$$

is not invertible.

LEMMA 2.3. *Assume the system in* (1.1) *satisfies the weaker stability condition and the equilibrium manifold can be expressed as*

$$\mathscr{E} = \{ U = E(u) : u \in \mathscr{D} \subset \mathbf{R}^{n-r} \}$$

with $E_u(u)$ *full-rank, as in* (1.3). *Then the following system of equations for* u:

$$P^I(E(u)) \left(E_u(u)\, u_t + \sum_{j=1}^{d} A_j(E(u))\, E_u(u)\, u_{x_j} \right) = 0,$$

is symmetrizable hyperbolic.

Proof. Since $Q(E(u)) \equiv 0$, we have $Q_U(E(u))\, E_u(u) \equiv 0$. Recall that P^I (resp. P^{II}) denotes the $(n-r) \times n$ (resp. $r \times n$) matrix consisting of the first $(n-r)$ (resp. last r) rows of $P(U_0)$. Because $PQ_U = \mathrm{diag}(0, \hat{S})\, P$ and $\hat{S}$ is invertible, we see that $P^{II}(E(u))\, E_u(u) = 0$. Moreover, $P^I(E(u))\, E_u(u)$ is invertible, since $E_u(u)$ and therefore $P(E(u))\, E_u(u)$ are of full-rank. Thus, the first $(n-r)$ columns of P^{-1} are $E_u(P^I E_u)^{-1}$.

On the other hand, we know from Theorem 2.2 that $P^{-*} A_0(U_0)\, P^{-1}$ is block-diagonal, say $P^{-*} A_0(U_0)\, P^{-1} = \mathrm{diag}(A_0^I, A_0^{II})$. Then $P^{-*} A_0 A_j P^{-1} = \mathrm{diag}(A_0^I, A_0^{II})\, P A_j P^{-1}$ is symmetric due to (ii). In particular, the upper-left corner $A_0^I P^I A_j E_u (P^I E_u)^{-1}$ is symmetric. Hence, $(P^I E_u)^*\, A_0^I$ symmetrizes the system under consideration. ∎

Next, we point out the relation between our stability condition and the *entropy condition* in [2]. In that paper, the authors considered the system in (1.1) with

$$CQ(U) \equiv 0 \tag{2.2}$$

for some constant $(n-r) \times n$ full-rank matrix C (*annihilator*). The system satisfies the entropy condition if there is a strictly convex smooth function $\Phi(U)$ such that

(1) $\Phi_{UU}(U)\, A_j(U)$ is symmetric for each j and $U \in G$;

(2) $\Phi_U(U)\, Q(U) \leqslant 0$ for $U \in G$;

(3) for any $U \in G$ the following three relations are equivalent

$$Q(U) = 0, \qquad \Phi_U(U)\, Q(U) = 0,$$

$$\Phi_U(U) = v^* C \qquad \text{for some} \quad (n-r)\text{-}vector\ v.$$

In addition, it is assumed in some way in [2] that the rank r of $Q_U(U)$ is equal to the number of its non-zero eigenvalues.

LEMMA 2.4. *Assume the rank of* $Q_U(U)$ *for* $U \in \mathscr{E}$ *is equal to the number of its non-zero eigenvalues. Then the entropy condition implies the weak stability condition.*

Proof. The assumption of the lemma immediately implies (i) with some $P(U)$. Let $\Phi(U)$ be the strictly convex entropy. Then $A_0(U) =: \Phi_{UU}(U)$ is symmetric positive definite and therefore (ii) is just (1).

To see the inequality in (iii)', we observe from (2) that $\Phi_U(U)\, Q(U)$ takes maximum values at $U \in \mathscr{E}$. Thus, at such U, the Hessian matrix is non-positive, that is,

$$\begin{aligned}
0 &\geqslant (\Phi_U Q)_{UU}(U) \\
&= \Phi_{UUU}(U)\, Q(U) + \Phi_{UU}(U)\, Q_U(U) + Q_U^*(U)\, \Phi_{UU}(U) \\
&\quad + \Phi_U(U)\, Q_{UU}(U) \\
&= A_0(U)\, Q_U(U) + Q_U^*(U)\, A_0(U) + v^* C Q_{UU}(U) \\
&= A_0(U)\, Q_U(U) + Q_U^*(U)\, A_0(U) + v^*(CQ)_{UU}(U) \\
&= A_0(U)\, Q_U(U) + Q_U^*(U)\, A_0(U).
\end{aligned}$$

Here we have used that $Q(U) = 0$, $\Phi_U(U) = v^* C$ due to (3) and the relation in (2.2). ▮

Now we point out an equivalent version of the stability condition for multi-dimensional 2×2-systems with $r = 1$:

$$\begin{aligned}
u_t + \sum_{j=1}^{d} f_j(u, v)_x &= 0, \\
v_t + \sum_{j=1}^{d} g_j(u, v)_x &= \frac{q(u, v)}{\varepsilon}.
\end{aligned} \tag{2.3}$$

Here u, v are both scalars and it is assumed that $q(u, v) = 0$ is equivalent to $v = h(u)$ with a given function h, as in (1.3). Moreover, $q_v(u, v) \neq 0$ for all (u, v) under consideration.

The relevant result can be stated as

LEMMA 2.5. *The system in* (2.3) *admits the stability condition if and only if* $q_v < 0$ *and there exist two positive functions* $\kappa_1(u, v)$ *and* $\kappa_2(u, v)$ *such that for all j,*

$$\kappa_1 q_v f_{jv} = \kappa_2 (q_v g_{ju} + q_u f_{ju} - q_u g_{jv} - q_u^2 q_v^{-1} f_{jv}). \tag{2.4}$$

Proof. Assume $q_v < 0$ and the existence of positive functions κ_1, κ_2 satisfying the relation in (2.4). Set $\alpha(u, v) = q_u(u, v)/q_v(u, v)$ and define

$$P_0(u, v) = \begin{pmatrix} 1 & 0 \\ \alpha(u, v) & 1 \end{pmatrix} \quad \text{and} \quad A_0 = P_0^* \begin{pmatrix} \kappa_1(u, v) & 0 \\ 0 & \kappa_2(u, v) \end{pmatrix} P_0.$$

Then

$$P_0 \begin{pmatrix} 0 & 0 \\ q_u & q_v \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & q_v \end{pmatrix} P_0 \quad \text{and} \quad A_0 \begin{pmatrix} f_{ju} & f_{jv} \\ g_{ju} & g_{jv} \end{pmatrix} = \left\{ A_0 \begin{pmatrix} f_{ju} & f_{jv} \\ g_{ju} & g_{jv} \end{pmatrix} \right\}^*.$$

Moreover,

$$A_0 \begin{pmatrix} 0 & 0 \\ q_u & q_v \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ q_u & q_v \end{pmatrix}^* A_0 = 2\kappa_2 q_v P_0^* \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} P_0 \leqslant 0.$$

Thus, the stability condition is satisfied.

Conversely, assume the stability condition is satisfied with some P and A_0. In view of Theorem 2.2, there exist $\kappa_1, \kappa_2 > 0$ such that

$$P^{-*} A_0 P^{-1} = \operatorname{diag}(\kappa_1, \kappa_2).$$

Here P may be assumed to be P_0. In fact, since Q_U has distinct eigenvalues 0 and $q_v (\neq 0)$, the general form of P satisfying (i) of the stability condition is $P = DP_0$ with D being an arbitrary invertible diagonal matrix. It follows from (iii) in the stability condition that $\kappa_2 q_v < 0$ and thereby $q_v < 0$. On the other hand, since $A_0 A_j$ are all symmetric with

$$A_j = \begin{pmatrix} f_{ju} & f_{jv} \\ g_{ju} & g_{jv} \end{pmatrix},$$

so are $P^{-*} A_0 P^{-1} P A_j P^{-1} = \operatorname{diag}(\kappa_1, \kappa_2)\, P A_j P^{-1}$. By writing out the explicit expression of $P A_j P^{-1}$, we see immediately that the relation in (2.4) is satisfied by κ_1 and κ_2. Hence, the proof is complete. ∎

We conclude this section by showing the equivalence of our stability condition and the subcharacteristic condition in [21, 14] for one-dimensional 2×2-systems with $r = 1$. Indeed, the subcharacteristic condition is that at $(u, v) = (u, h(u))$, $q_v < 0$ and

$$\det \begin{pmatrix} f_u - \lambda^* & f_v \\ g_u & g_v - \lambda^* \end{pmatrix} = (f_u - \lambda^*)(g_v - \lambda^*) - f_v g_u < 0. \tag{2.5}$$

Here $\lambda^* = f_u(u, h(u)) - \alpha(u, h(u))\, f_v(u, h(u))$ is the characteristics of the equilibrium system. If the subcharacteristic condition holds, then $\kappa_1(u, v) \equiv f_v(g_u + \alpha f_u - \alpha g_v - \alpha^2 f_v) > 0$ at $(u, v) = (u, h(u))$. Taking $\kappa_2(u, v) = f_v^2$, we see that the relation in (2.4) is satisfied by κ_1 and κ_2 so chosen. Since the sign of $\kappa_1(u, v)$, $\kappa_2(u, v)$ and $-q_v(u, v)$ does not change for (u, v) close to $(u, h(u))$, our stability condition is verified. Conversely, we only need to notice that the relation in (2.4) implied that in (2.5).

3. EXAMPLES AND APPLICATIONS

This section is devoted to some important physical examples satisfying our stability condition. We will often use Theorem 2.2.

3.1. *Equations in the Kinetic Theory of Gases*

Discrete velocity models of the Boltzmann equation in the kinetic theory of gases have the form

$$U_t + \sum_{j=1}^{d} A_j U_{x_j} = \frac{Q(U)}{\varepsilon}. \tag{3.1}$$

Here U is an n-vector with components being the density functions; $A_j (j = 1, 2, ..., d)$ are (constant) diagonal matrices; $Q(U)$, known as the collision operator, is an n-vector with quadratic forms of U as components; ε is proportional to the mean free path of particles under consideration. For details we refer to [18].

On physical grounds, the solution components of (3.1) are non-negative. As in [1] which treats the one-dimensional Broadwell model, we assume that the solution components are positive. Thus, the state space G is taken to be

$$G = \{ U \in \mathbf{R}^n : u_k > 0 \text{ for } k = 1, 2, ..., n \},$$

where u_k denotes the kth component of U.

For lots of models widely studied, such as Carleman models, Broadwell models, etc., $Q_U(U)$ can be decomposed as

$$Q_U(U) = A(U)\, A_0(U), \tag{3.2}$$

where $A(U)$ is a negative semidefinite Hermitian matrix and $A_0(U)$ is a diagonal matrix.

Obviously, $A_0(U)$ symmetrizes the A_j's. Furthermore, we can prove that if $A_0(U)$ is positive definite, then the stability condition is satisfied. Indeed, we have

LEMMA 3.1. *Let (3.2) hold with A, A_0 being negative semidefinite Hermitian and positive definite, respectively. Then there is an invertible matrix P such that*

$$PQ_U = \Lambda P \qquad \text{and} \qquad A_0 Q_U = P^* \Lambda P,$$

where Λ is a diagonal matrix with non-positive entries.

Proof. Since $A_0^{1/2} A A_0^{1/2}$ is Hermitian, there is a unitary matrix H such that $A_0^{1/2} A A_0^{1/2} = H^* \Lambda H$ with Λ being a diagonal matrix. Furthermore, Λ is negative semidefinite since so is A and A_0 is positive definite. Take $P = H A_0^{1/2}$. Then

$$PQ_U = H A_0^{1/2} A A_0^{1/2} A_0^{1/2} = HH^* \Lambda H A_0^{1/2} = \Lambda P,$$
$$A_0 Q_U = A_0^{1/2} A_0^{1/2} A A_0^{1/2} A_0^{1/2} = A_0^{1/2} H^* \Lambda H A_0^{1/2} = P^* \Lambda P.$$

This completes the proof. ∎

Here are several examples. We denote by Q_k the k-the component of $Q(U)$.

The one-dimensional Carleman model. $n = 2$ and $Q(U) = (u_2^2 - u_1^2, u_1^2 - u_2^2)^*$. A direct calculation shows

$$Q_U(U) = \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix} \equiv A(U)\, A_0(U).$$

Obviously, $A(U)$ is symmetric negative semidefinite and $A_0(U)$ is positive definite for $U \in G$.

The three-dimensional Broadwell model. $n = 6$ and

$$Q_1(U) = Q_2(U) = u_3 u_4 + u_5 u_6 - 2u_1 u_2,$$
$$Q_3(U) = Q_4(U) = u_1 u_2 + u_5 u_6 - 2u_3 u_4,$$
$$Q_5(U) = Q_6(U) = u_1 u_2 + u_3 u_4 - 2u_5 u_6.$$

Set $X = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Then

$$Q_U(U) = \begin{pmatrix} -2X & X & X \\ X & -2X & X \\ X & X & -2X \end{pmatrix} \operatorname{diag}(u_2, u_1, u_4, u_3, u_6, u_5) \equiv A(U)\, A_0(U).$$

It is easy to verify that the eigenvalues of the symmetric matrix $A(U)$ are 0 and -6 with the multiplicity 4 and 2, respectively. And $A_0(U)$ is positive definite for $U \in G$.

The one-dimensional coplanar model. $n=4$, $Q_1(U)=Q_2(U)=u_3u_4-u_1u_2$ and $Q_3(U)=Q_4(U)=-Q_1(U)$. With the X defined above, we see

$$Q_U(U)=\begin{pmatrix} -X & X \\ X & -X \end{pmatrix} \operatorname{diag}(u_2, u_1, u_4, u_3) \equiv A(U)\, A_0(U).$$

It is easy to verify that the eigenvalues of the symmetric matrix $A(U)$ are -4 and 0 with the multiplicity 3. And $A_0(U)$ is positive definite for $U \in G$.

Similarly we can show that the one-, two-dimensional Broadwell models both have the structure in (3.2). Furthermore, it is not difficult to see that all the above models admit the assumption in (1.3) and Lemma 2.2.

3.2. *A Simplified Model for Reacting Flow*

In order to develop numerical methods for chemically reacting flow problems, LeVeque and others in [13] (see also [19]) presented the following simplified mathematical model for reacting flow

$$\begin{aligned} \rho_t + u\rho_x + \rho u_x &= 0, \\ u_t + uu_x + \rho^{-1} p_x &= 0, \\ s_t + us_x &= \frac{s_e(\rho) - s}{\varepsilon}, \end{aligned} \tag{3.3}$$

where ρ and u are the respective density and fluid velocity, s is the mass fracion of one mode of a two-mode gas, $p = p(\rho, s)$ is the pressure, and $s_e(\rho)$ is a given equilibrium distribution function of ρ.

Here we show that this model system admits the stability condition. To this end, set $U = (\rho, u, s)^*$ and $Q(U) = (0, 0, s_e(\rho) - s)^*$. Then we have

$$U_t + A_1(U)\, U_x = \frac{Q(U)}{\varepsilon}$$

with

$$A_1(U) = \begin{pmatrix} u & \rho & 0 \\ p_\rho \rho^{-1} & u & p_s \rho^{-1} \\ 0 & 0 & u \end{pmatrix}.$$

Since

$$Q_U(U) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ s_{e\rho} & 0 & -1 \end{pmatrix},$$

its rank $r(=1)$ is equal to the number of its non-zero eigenvalues. Thus, (i) of the stability condition is satisfied with some P. By Theorem 2.2, it suffices to find a positive definite symmetrizer A_0 such that (iii)' of the weaker stability condition holds, that is,

$$A_0 Q_U + Q_U^* A_0 \leqslant 0.$$

To do so, let $s_{ep} \equiv ds_e/d\rho \neq 0$ and define

$$A_0(U) = \begin{pmatrix} p_\rho & 0 & p_s \\ 0 & \rho^2 & 0 \\ p_s & 0 & \dfrac{-p_s}{s_{ep}} \end{pmatrix}.$$

Obviously, $A_0(U)$ is symmetric. Moreover, if $\rho \neq 0$, $p_\rho + p_s s_{ep} > 0$ and $p_s s_{ep} < 0$, then $A_0(U)$ is positive definite. Thus, we take

$$G = \{ U \in \mathbf{R}^3 \colon \rho \neq 0,\ p_\rho(\rho, s) > -p_s(\rho, s)\, s_{ep}(\rho) > 0 \}. \tag{3.4}$$

A simple calculation indicates that $A_0 A_1$ is symmetric and (iii)' is satisfied. Hence the above model system admits the stability condition.

It is shown in [23] that the constraint $U \in G$ guarantees the asymptotic stability of relaxation shock profiles for the above model system. Note that $p_\rho + p_s s_{ep} > 0$ implies that the corresponding equilibrium system

$$\begin{aligned} \rho_t + u\rho_x + \rho u_x &= 0, \\ u_t + uu_x + \rho^{-1} p_x &= 0, \\ s &= s_e(\rho) \end{aligned}$$

is strictly hyperbolic.

In [13], the pressure function is taken to be $p = \rho c^2(1 - s + \beta s)$ with c and β being constants. Thus,

$$G = \{ U \in \mathbf{R}^3 \colon \rho \neq 0,\ 1 - s + \beta s > \rho(1-\beta)\, s_{ep}(\rho) > 0 \}.$$

The constraint $\rho(1-\beta)\, s_{ep}(\rho) > 0$ is also derived in [13] but with a different approach.

For the system in (3.3), it is obvious that the corresponding limiting inner system

$$\tilde{\rho}_\tau = 0, \qquad \tilde{u}_\tau = 0, \quad \tilde{s}_\tau = s_e(\tilde{\rho}) - \tilde{s}$$

has unique continuous solutions whose limits at infinity exist.

3.3. *Relativistic Magnetohydrodynamics*

For a one-dimensional conducting fluid in a magnetic field, the usual equations of mass, momentum and energy in fluid dynamics are modified by the inclusion of the magnetic force in the momentum equation and the Joule heat term in the entropy equation. These, together with Maxwell's equations and Ohm's law, give the following system (see [21])

$$\begin{aligned}
\rho_t + u\rho_x + \rho u_x &= 0,\\
u_t + uu_x + \rho^{-1} p_x &= \frac{B(E-uB)}{\varepsilon\rho},\\
p_t + up_x + \gamma p u_x &= \frac{(\gamma-1)(E-uB)^2}{\varepsilon},\\
B_t + E_x &= 0,\\
E_t + c^2 B_x &= \frac{-\kappa(E-uB)}{\varepsilon},
\end{aligned} \tag{3.5}$$

where ρ, p, and u are the respective fluid density, pressure and velocity, B is the magnetic induction, and E is the electric field; γ is the ratio of specific heats, c is the velocity of light, $1/\kappa$ is the dielectric constant, and $1/\varepsilon$ is the electric conductivity.

Unlike Whitham in [21], we consider the relativistic effect and therefore do not omit the displacement current E_t/κ.

Now we show that the above system satisfies our stability condition. To this end, set $U = (\rho, u, p, B, E)^*$ and $Q(U) = (0, \rho^{-1}B, (\gamma-1)(E-uB), 0, -\kappa)^*\,(E-uB)$. Then we have

$$U_t + A_1(U)\, U_x = \frac{Q(U)}{\varepsilon}$$

with

$$A_1(U) = \begin{pmatrix} u & \rho & 0 & 0 & 0\\ 0 & u & \rho^{-1} & 0 & 0\\ 0 & \gamma p & u & 0 & 0\\ 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & c^2 & 0 \end{pmatrix}.$$

Since for $U \in \mathscr{E}$,

$$Q_U(U) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -\rho^{-1}B^2 & 0 & -\rho^{-1}Bu & \rho^{-1}B \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \kappa B & 0 & \kappa u & -\kappa \end{pmatrix},$$

its rank $r(=1)$ is equal to the number of its non-zero eigenvalues. Thus, (i) is satisfied with

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & \rho\kappa & 0 & 0 & B \\ 0 & \sqrt{\kappa}\, B & 0 & \sqrt{\kappa}\, u & -\sqrt{\kappa} \end{pmatrix}. \tag{3.6}$$

By Theorem 2.2, it suffices to find a positive definite symmetrizer A_0 such that (iii)' holds. To do so, define

$$A_0(U) = \begin{pmatrix} 1 & 0 & -\dfrac{\rho}{\gamma p} & 0 & 0 \\ 0 & \kappa\rho & 0 & 0 & 0 \\ -\dfrac{\rho}{\gamma p} & 0 & \dfrac{\rho^2}{\gamma^2 p^2} + \dfrac{\kappa}{\gamma p} & 0 & 0 \\ 0 & 0 & 0 & c^2 & -u \\ 0 & 0 & 0 & -u & 1 \end{pmatrix}.$$

Obviously, $A_0(U)$ is symmetric. By a direct calculation, we know that $A_0(U)$ is positive definite and $A_0(U)\, A_1(U)$ is symmetric for U in

$$G = \{ U \in \mathbf{R}^5 : |u| < c,\ p > 0,\ \rho > 0 \}. \tag{3.7}$$

Moreover, $A_0 Q_U + Q_U^* A_0 \leqslant 0$. Hence the stability condition is satisfied.

For the system in (3.5), the corresponding limiting inner system is

$$\begin{aligned} &\tilde{\rho}_\tau = 0, \qquad \tilde{u}_\tau = \tilde{\rho}^{-1}\tilde{B}(\tilde{E} - \tilde{u}\tilde{B}), \quad \tilde{p}_\tau = (\gamma - 1)(\tilde{E} - \tilde{u}\tilde{B})^2, \\ &\tilde{B}_\tau = 0, \qquad \tilde{E}_\tau = -\kappa(\tilde{E} - \tilde{u}\tilde{B}). \end{aligned}$$

Obviously, $\tilde{\rho}$ and $\tilde{B}$ are independent of τ. Thus we deduce from the second and the last equations that

$$(\tilde{E} - \tilde{u}\tilde{B})_\tau = -(\kappa + \tilde{\rho}^{-1}\tilde{B}^2)(\tilde{E} - \tilde{u}\tilde{B}).$$

Therefore, $(\tilde{E}-\tilde{u}\tilde{B})(\tau)$ decays exponentially to zero as τ goes to infinity since $\kappa+\tilde{\rho}^{-1}\tilde{B}^2 \geqslant \kappa > 0$. Substituting $(\tilde{E}-\tilde{u}\tilde{B})(\tau)$ into the second, the third and the last equations, we easily know that the limits of $\tilde{u}(\tau)$, $\tilde{p}(\tau)$ and $\tilde{E}(\tau)$ exist as τ goes to infinity and they converge exponentially to the limits.

By using the explicit expression of P in (3.6), we can write out the corresponding equilibrium system as

$$\begin{aligned}
\rho_t + u\rho_x + \rho u_x &= 0,\\
B_t + uB_x + Bu_x &= 0,\\
p_t + up_x + \gamma p u_x &= 0,\\
\rho\kappa(u_t + uu_x + \rho^{-1}p_x) + B(E_t + c^2 B_x) &= 0,\\
E &= uB.
\end{aligned}$$

Note that there is no a constant annihilator $C \in \mathbf{R}^{4\times 5}$ such that

$$CQ(U) \equiv 0.$$

Such an assumption is required in [2].

3.4. *Inviscid Gas Dynamics with Relaxation*

In changing flow the internal energy may lag behind the equilibrium value corresponding to the ambient pressure and density. This is the so-called relaxation effect and the equations of motion take the following form (see [21])

$$\begin{aligned}
&\rho_t + u\rho_x + \rho u_x = 0,\\
&u_t + uu_x + \rho^{-1}p_x = 0,\\
&p_t + up_x + \gamma p u_x = \frac{(\gamma-1)\,\rho(E-\mu p\rho^{-1})}{\varepsilon},\\
&\qquad E_t + uE_x = -\frac{E-\mu p\rho^{-1}}{\varepsilon},
\end{aligned} \tag{3.8}$$

where ρ, p, and u are the respective fluid density, pressure and velocity, and E is the energy in the lagging degrees of freedom; $\gamma > 1$ and $\mu > 0$ are two constants related to the degrees of freedom, and $1/\varepsilon$ is the relaxation time.

Here we show that this relaxation system admits the stability condition. To this end, set $U = (\rho, u, p, E)^*$ and $Q(U) = (0, 0, (\gamma-1)\,\rho, -1)^*$ $(E-\mu p\rho^{-1})$. Then we have

$$U_t + A_1(U)\,U_x = \frac{Q(U)}{\varepsilon}$$

with

$$A_1(U) = \begin{pmatrix} u & \rho & 0 & 0 \\ 0 & u & \rho^{-1} & 0 \\ 0 & \gamma p & u & 0 \\ 0 & 0 & 0 & u \end{pmatrix}.$$

Since for $U \in \mathscr{E}$,

$$Q_U(U) = \frac{1}{\rho^2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ (\gamma-1)\,\mu\rho p & 0 & -(\gamma-1)\,\mu\rho^2 & (\gamma-1)\,\rho^3 \\ -\mu p & 0 & \mu\rho & -\rho^2 \end{pmatrix},$$

its rank $r(=1)$ is equal to the number of its non-zero eigenvalues. Thus, (i) is satisfied with

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \rho(\gamma-1) \\ \mu\rho^{-1}p & 0 & -\mu & \rho \end{pmatrix}. \tag{3.9}$$

By Theorem 2.2, it suffices to find a positive definite symmetrizer A_0 such that (iii)' holds. To do so, define

$$A_0(U) = \frac{1}{(\gamma-1)\,\rho^2} \begin{pmatrix} \gamma\mu p^2 & 0 & -\mu\rho p & 0 \\ 0 & \mu(\gamma-1)\,\rho^3 p & 0 & 0 \\ -\mu\rho p & 0 & \mu\rho^2 & 0 \\ 0 & 0 & 0 & (\gamma-1)\,\rho^4 \end{pmatrix}.$$

Obviously, $A_0(U)$ is symmetric. By a direct calculation, we know that $A_0(U)$ is positive definite and $A_0(U)\,A_1(U)$ is symmetric for U in

$$G = \{U \in \mathbf{R}^4 \colon p > 0, \rho > 0\}. \tag{3.10}$$

Moreover, $A_0 Q_U + Q_U^* A_0 \leqslant 0$. Hence the stability condition is satisfied.

For the system in (3.8), the corresponding limiting inner system is

$$\tilde{\rho}_\tau = 0, \qquad \tilde{u}_\tau = 0, \quad \tilde{p}_\tau = (\gamma-1)(\tilde{\rho}\widetilde{E} - \mu\tilde{p}), \quad \widetilde{E}_\tau = -(\widetilde{E} - \mu\tilde{\rho}^{-1}\tilde{p}).$$

Obviously, $\tilde{\rho}$ and $\tilde{u}$ are independent of τ. Thus we deduce from the last two equations that

$$(\widetilde{E} - \mu\tilde{\rho}^{-1}\tilde{p})_\tau = -(1 + \mu(\gamma-1))(\widetilde{E} - \mu\tilde{\rho}^{-1}\tilde{p}).$$

Therefore, $(\tilde{E}-\mu\tilde{\rho}^{-1}\tilde{p})(\tau)$ decays exponentially to zero as τ goes to infinity. Substituting $(\tilde{E}-\mu\tilde{\rho}^{-1}\tilde{p})(\tau)$ into the last two equations, we easily know that the limits of $\tilde{p}(\tau)$ and $\tilde{E}(\tau)$ exist as τ goes to infinity and they converge exponentially to the limits.

By using the explicit expression of P in (3.9), we can write out the corresponding equilibrium system as

$$\rho_t + u\rho_x + \rho u_x = 0,$$

$$u_t + uu_x + \rho^{-1}p_x = 0,$$

$$p_t + up_x + \frac{\gamma+\mu(\gamma-1)}{1+\mu(\gamma-1)}\, pu_x = 0,$$

$$E = \mu\rho^{-1}p.$$

4. FORMAL ASYMPTOTIC APPROXIMATIONS

In this section we construct formal asymptotic approximations of the initial-layer solution U^{ε} to the nonlinear problem in (1.1) by a variant of the classical matched expansion method developed in [17]. This variant method has been successfully applied to our problem (1.1) with $d=0$, that is, systems of ordinary differential equations, in [17, Chap. 4; 6].

The idea in [17] can be explained as follows. One starts by seeking a solution (so-called *outer expansion*) of the form

$$U(x, t, \varepsilon) \sim \sum_{k=0}^{\infty} \varepsilon^k U_k(x, t). \tag{4.1}$$

This first step is natural because our problem involves a small parameter ε. However, it will be seen immediately that such a solution cannot generally satisfy the prescribed initial conditions. Thus, one attempts to correct the outer expansion by adding an *initial-layer correction*

$$I(x, t, \varepsilon) \sim \sum_{k=0}^{\infty} \varepsilon^k I_k(x, \tau) \tag{4.2}$$

with *inner variable* $\tau = t/\varepsilon$ near $t=0$. As a correction, $I(x, t, \varepsilon)$ will be significant only near $t=0$. Thus the $I_k(x, \tau)$'s are required to decay to zero as τ goes to infinity, since the latter happens as ε tends to zero whenever $t \geqslant \delta > 0$ with δ arbitrary but fixed. This natural requirement is similar to the traditional *matching principle* in [4]. Once the outer expansion and the initial-layer correction are found, the formal asymptotic approximation is defined as the following truncation with a fixed m:

$$U_\varepsilon^m = \sum_{k=0}^{m} \varepsilon^k U_k(x, t) + \sum_{k=0}^{m} \varepsilon^k I_k(x, \tau). \tag{4.3}$$

For further discussions, we recall the equations in (1.1) and introduce an operator R, acting on $V = V(x, t) \in \mathbf{R}^n$, defined as

$$R(V) := V_t + \sum_{j=1}^{d} A_j(V)\, V_{x_j} - \frac{Q(V)}{\varepsilon}.$$

In addition, we will often use formal asymptotic expansions like

$$A\left(\sum_{k=0}^{\infty} \varepsilon^k V_k\right) = A(V_0) + \sum_{k=1}^{\infty} \varepsilon^k [A_U(V_0)\, V_k + \mathscr{C}(A, k, \underline{V})].$$

Note that coefficients $\mathscr{C}(A, k, \underline{V})$ are completely determined by the given function A and the first k components $V_0, V_1, ..., V_{k-1}$ of $\underline{V} := (V_0, V_1, V_2, ...)$. Moreover, $\mathscr{C}(A, 1, \underline{V}) = 0$ and $\mathscr{C}(A, k, \underline{V})$ is linear with respect to V_{k-1} for $k \geqslant 3$.

In the following Subsections 4.1 and 4.2, we will show how to find the expansions in (4.1) and (4.2). Subsection 4.3 is devoted to the formal asymptotic approximation in (4.3).

4.1. *Outer Expansions*

As a formal solution, the outer expansion asymptotically satisfies the equations. Namely, the formal asymptotic expansion

$$\begin{aligned} R\left(\sum_{k=0}^{\infty} \varepsilon^k U_k\right) = {} & -\varepsilon^{-1} Q(U_0) + \sum_{k=0}^{\infty} \varepsilon^k \\ & \times \Big\{ U_{kt} + \sum_j A_j(U_0)\, U_{kx_j} - Q_U(U_0)\, U_{k+1} \\ & + \sum_j \sum_{h=1}^{k} [A_{jU}(U_0)\, U_h + \mathscr{C}(A_j, h, \underline{U})] \\ & \times U_{k-h, x_j} - \mathscr{C}(Q, k+1, \underline{U}) \Big\} \end{aligned} \tag{4.4}$$

vanishes. This happens when each term of the last expansion is zero, i.e.,

$$Q(U_0) = 0, \tag{4.5}$$

$$U_{kt} + \sum_j A_j(U_0)\, U_{kx_j} = Q_U(U_0)\, U_{k+1} + \mathscr{C}'(k+1, \underline{U}), \qquad k \geqslant 0. \tag{4.6}$$

Here $\mathscr{C}'(k+1, \underline{U}) = \mathscr{C}(Q, k+1, \underline{U}) - \sum_j \sum_{h=1}^{k} [A_{jU}(U_0)\, U_h + \mathscr{C}(A_j, h, \underline{U})]$ U_{k-h, x_j} is completely determined by $U_0, U_1, ..., U_k$.

Obviously, the equations in (4.6) need to be rewritten to determine U_k inductively. Equation (4.5) shows that U_0 lies on the equilibrium manifold $\mathscr{E}$, which indicates that the outer expansion cannot take the initial value $\bar{U}(x, \varepsilon)$ if $Q(\bar{U}(x, 0)) \neq 0$. The stability condition says that an invertible matrix $P = P(U_0)$ is well defined on the equilibrium manifold such that

$$PQ_U(U_0) = \begin{pmatrix} 0 & 0 \\ 0 & \hat{S}(U_0) \end{pmatrix} P.$$

In view of this, we multiply the equations in (4.6) with P from the left to obtain

$$P^I(U_0) \left(U_{kt} + \sum_{j=1}^{d} A_j(U_0)\, U_{kx_j} \right) = P^I(U_0)\, \mathscr{C}'(k+1, \underline{U}), \tag{4.7}$$

$$P^{II}(U_0) \left(U_{kt} + \sum_{j=1}^{d} A_j(U_0)\, U_{kx_j} \right) = \hat{S}(U_0)\, P^{II}(U_0)\, U_{k+1} + P^{II}(U_0)\, \mathscr{C}'(k+1, \underline{U}). \tag{4.8}$$

Here, as in Lemma 2.3, P^I and P^{II} denote the respective matrices composed of the first $(n-r)$ and last r rows of P.

The leading term U_0 is obtained by solving the equilibrium system (4.5) and (4.7) with $k = 0$. From the assumption in (1.3) that the equilibrium manifold $\mathscr{E}$ can be written as

$$\mathscr{E} = \{ U = E(u) : u \in \mathscr{D} \subset \mathbf{R}^{n-r} \},$$

$U_0 = E(u)$ follows from (4.5). Substituting $U_0 = E(u)$ into (4.7) with $k = 0$ (Note $\mathscr{C}'(1, \underline{U}) = 0$) yields

$$P^I(E(u)) \left(E_u(u)\, u_t + \sum_{j=1}^{d} A_j(E(u))\, E_u(u)\, u_{x_j} \right) = 0. \tag{4.9}$$

As shown in Lemma 2.3, this is a quasilinear symmetrizable hyperbolic system for u. Thus, the standard existence theory in [15] can be applied to conclude the existence of a local (in time) classical solution u provided that $u(x, 0)$ is given appropriately. As a consequence, U_0 is determined with $U_0 = E(u)$.

Concerning U_k with $k \geqslant 1$, we note that $P^{II} U_k$ can be determined algebraically from (4.8) when $U_h (h = 0, 1, ..., k-1)$ are known, since $\hat{S}(U_0)$ is invertible. Thus, it remains to find equations for $P^I U_k$. Under the stability condition, the linear ($k \geqslant 2$) or semilinear ($k = 1$) system in (4.7) can be converted to a symmetrizable hyperbolic one for $P^I U_k$. In fact, (4.6) for U_k can be symmetrized with $A_0(U_0)$. Thus, PU_k satisfies a symmetrizable hyperbolic system with $PQ_U(U_0)\, U_{k+1}$ unknown. Recall from Theorem 2.2 that $P^{-*} A_0(U_0)\, P^{-1}$ is block-diagonal. It is easy to see that (4.7) is a symmetrizable hyperbolic system for $P^I U_k$ if $P^{II} U_k$ is provided as above. By such a procedure, the U_k's will be completely determined provided that we specify initial values $U_0(x, 0) \in \mathscr{E}$ and $P^I U_k(x, 0)$ with $k \geqslant 1$.

4.2. *Composite Expansions*

In order to find these initial values $U_0(x, 0)$ and $P^I U_k(x, 0) (k \geqslant 1)$, we turn to consider the *composite expansion*

$$\sum_{k=0}^{\infty} \varepsilon^k U_k(x, t) + \sum_{k=0}^{\infty} \varepsilon^k I_k(x, \tau)$$

defined in (4.1) and (4.2). As a corrected formal solution of the problem in (1.1), this composite expansion should take the prescribed initial value $\bar{U}(x, \varepsilon)$, i.e.,

$$\sum_{k=0}^{\infty} \varepsilon^k U_k(x, 0) + \sum_{k=0}^{\infty} \varepsilon^k I_k(x, 0) \sim \bar{U}(x, \varepsilon).$$

Let $\bar{U}(x, \varepsilon)$ have a formal expansion $\bar{U}(x, \varepsilon) \sim \sum_{k=0}^{\infty} \varepsilon^k \bar{U}_k(x)$. Then we take, for each k,

$$U_k(x, 0) + I_k(x, 0) = \bar{U}_k(x). \tag{4.10}$$

This is not enough to determine $U_k(x, 0)$, since I_k is unknown.

However, the composite expansion allows an analysis like that in (4.4)–(4.6) for the outer expansion. To make this analysis, we replace t with $\varepsilon\tau$ in the coefficients of the outer expansion and formally expand them at $\varepsilon = 0$:

$$U_k(x, t) = U_k(x, \varepsilon\tau) = \sum_{h=0}^{\infty} \frac{\varepsilon^h \tau^h}{h!} \frac{\partial^h U_k}{\partial t^h}(x, 0).$$

With these, the outer expansion becomes

$$\sum_{k=0}^{\infty} \varepsilon^k U_k(x, \varepsilon\tau) = \sum_{k=0}^{\infty} \varepsilon^k \sum_{h=0}^{\infty} \frac{\varepsilon^h \tau^h}{h!} \frac{\partial^h U_k}{\partial t^h}(x, 0) \equiv \sum_{k=0}^{\infty} \varepsilon^k P_k(x, \tau),$$

where

$$P_k(x, \tau) = \sum_{h=0}^{k} \frac{\tau^h}{h!} \frac{\partial^h U_{k-h}}{\partial t^h}(x, 0).$$

The composite expansion becomes

$$\sum_{k=0}^{\infty} \varepsilon^k [P_k(x, \tau) + I_k(x, \tau)],$$

which is just the traditional *inner expansion* (see [4]). The corrected formal solution should asymptotically satisfy the equations in (1.1). Namely, the formal asymptotic expansion

$$\begin{aligned}
R&\left(\sum_{k=0}^{\infty} \varepsilon^k U_k(x, \varepsilon\tau) + \sum_{k=0}^{\infty} \varepsilon^k I_k(x, \tau)\right) \\
&= R\left(\sum_{k=0}^{\infty} \varepsilon^k [P_k(x, \tau) + I_k(x, \tau)]\right) \\
&= \varepsilon^{-1}[(P_0 + I_0)_\tau - Q(P_0 + I_0)] \\
&\quad + \sum_{k=1}^{\infty} \varepsilon^{k-1} \Big\{ (P_k + I_k)_\tau - Q_U(P_0 + I_0)(P_k + I_k) - \mathscr{C}(Q, k, \underline{P+I}) \\
&\quad + \sum_j A_j(P_0 + I_0)(P_{k-1} + I_{k-1})_{x_j} \\
&\quad + \sum_j \sum_{h=1}^{k-1} [A_{jU}(P_0 + I_0)(P_h + I_h) + \mathscr{C}(A_j, h, \underline{P+I})] \\
&\quad \times (P_{k-h-1} + I_{k-h-1})_{x_j} \Big\}
\end{aligned} \tag{4.11}$$

vanishes. This happens when each term of the last expansion is zero, i.e.,

$$\begin{aligned}
&(P_0 + I_0)_\tau = Q(P_0 + I_0), \\
&(P_k + I_k)_\tau = Q_U(P_0 + I_0)(P_k + I_k) + \mathscr{C}''(k, \underline{P+I}), \qquad k \geqslant 1.
\end{aligned} \tag{4.12}$$

Here

$$\mathscr{C}''(k, \underline{P+I}) = \mathscr{C}(Q, k, \underline{P+I}) - \sum_j A_j(P_0+I_0)(P_{k-1}+I_{k-1})_{x_j}$$
$$-\sum_j \sum_{h=1}^{k-1} [A_{jU}(P_0+I_0)(P_h+I_h) + \mathscr{C}(A_j, h, \underline{P+I})]$$
$$\times (P_{k-h-1}+I_{k-h-1})_{x_j}$$

is completely determined by $P_0+I_0, P_1+I_1, ..., P_{k-1}+I_{k-1}$.

Similarly, we have

$$\begin{aligned} &P_{0\tau} - Q(P_0) = 0, \\ &P_{k\tau} - Q_U(P_0)\, P_k = \mathscr{C}''(k, P), \qquad k \geqslant 1, \end{aligned} \tag{4.13}$$

because the outer expansion formally solves the equations, i.e.,

$$R\left(\sum_{k=0}^{\infty} \varepsilon^k P_k(x, \tau)\right) = R\left(\sum_{k=0}^{\infty} \varepsilon^k U_k(x, t)\right) = 0.$$

Setting

$$\tilde{I}_k(x, \tau) = I_k(x, \tau) + U_k(x, 0)$$

and noting $P_0 = U_0(x, 0)$, we find from (4.12) and (4.13) that

$$\begin{aligned} &\frac{d\tilde{I}_0}{d\tau} = \frac{dI_0}{d\tau} = Q(\tilde{I}_0(x, \tau)), \\ &\frac{d\tilde{I}_k}{d\tau} = Q_U(\tilde{I}_0)\, \tilde{I}_k - Q_U(U_0(x, 0))\, U_k(x, 0) + r_k(x, \tau) \end{aligned} \tag{4.14}$$

with $k \geqslant 1$ and

$$r_k(x, \tau) = [Q_U(\tilde{I}_0) - Q_U(U_0(x, 0))](P_k - U_k(x, 0))$$
$$+ \mathscr{C}''(k, \underline{P+I}) - \mathscr{C}''(k, \underline{P}).$$

Up to now, we have obtained equations in (4.14) and initial conditions in (4.10) for $\tilde{I}_k(x, \tau)$. By solving the initial value problems of ordinary differential Eqs. (4.14) with (4.10), we will get $\tilde{I}_k(x, \tau)$. Recall that $\tilde{I}_k(x, \tau)$ is defined as $I_k(x, \tau) + U_k(x, 0)$. The aforementioned initial values $U_k(x, 0)$ are taken to be

$$U_k(x, 0) = \tilde{I}_k(x, \infty)$$

due to the matching principle that $I_k(x, \tau)$ decays to zero as τ goes to infinity. Clearly, we need to show the existence of $\tilde{I}_k(x, \infty)$, which will be done in the next section.

Next we describe a procedure to determine the coefficients of the expansions in (4.1) and (4.2). The first equation in (4.14) with $\tilde{I}_0(x, 0) = \bar{U}_0(x)$ in (4.10) is just the *limiting inner problem*, which will be assumed to have a unique solution converging to $\tilde{I}_0(x, \infty) \in \mathscr{E}$ as τ goes to infinity. Because of the macthing principle, we take $U_0(x, 0) = \tilde{I}_0(x, \infty)$. Knowing $U_0(x, 0)$, we set $I_0(x, \tau) = \tilde{I}_0(x, \tau) - U_0(x, 0)$ and solve the reduced problem (4.5) and (4.7) with $k = 0$ with the initial value $U_0(x, 0)$ to obtained $U_0(x, t)$. Moreover, $P^{II} U_1$ is also obtained from (4.8) with $k = 0$. Assume that U_h, I_h with $h < k$ have been obtained. Then $\mathscr{C}'(k, \underline{U})$ in (4.8) and $r_k(x, \tau)$ in (4.14) are completely determined. Moreover, $P^{II} U_k$ can be solved algebraically from (4.8) and therefore $Q_U(U_0)\, U_k = P^{-1} \operatorname{diag}(0, \hat{S}(U_0))\, PU_k$ in (4.14) is known. Now, we solve the equations in (4.14) with $\tilde{I}_k(x, 0) = \bar{U}_k(x)$ in (4.10). It will be shown in the next section that $\tilde{I}_k(x, \tau)$ exists uniquely, decays exponentially to $\tilde{I}_k(x, \infty)$ and $P^{II}(U_0(x, 0))\, \tilde{I}_k(x, \infty) = P^{II}(U_0(x, 0))\, U_k(x, 0)$. Because of the matching principle, we take $(P^I U_k)(x, 0) = P^I(U_0(x, 0))\, \tilde{I}_k(x, \infty)$ and solve the symmetrizable hyperbolic system in (4.7) to get $P^I U_k$. In conclusion, we have determined all coefficients in expansions (4.1) and (4.2).

4.3. *Formal Asymptotic Approximations*

Finally, we turn to the formal asymptotic approximations U_ε^m in (4.3). Because the U_k's satisfy the equations in (4.4)–(4.6), it is clear that $\sum_{k=0}^m \varepsilon^k U_k$ is a formal approximation to the solution of the equations with the residual

$$R\left(\sum_{k=0}^{m} \varepsilon^k U_k(x, t)\right) = \varepsilon^m Q_U(U_0)\, U_{m+1} + O(\varepsilon^{m+1}). \tag{4.15}$$

Recall from (4.6) that the coefficient of ε^m

$$Q_U(U_0)\, U_{m+1} = U_{mt} + \sum_j A_j(U_0)\, U_{mx_j} - \mathscr{C}'(m+1, \underline{U})$$

is completely determined by $U_0, U_1, ..., U_m$. Define F_m by

$$\varepsilon^m F_m = R(U_\varepsilon^m) - \varepsilon^m Q_U(U_0)\, U_{m+1}.$$

Note that $F_m = F_m(x, \tau, \varepsilon)$ depends on x, τ, ε and $t = \varepsilon\tau$. Then we have

$$U_{\varepsilon t}^m + \sum_j A_j(U_\varepsilon^m)\, U_{\varepsilon x_j}^m = \frac{Q(U_\varepsilon^m)}{\varepsilon} + \varepsilon^m Q_U(U_0)\, U_{m+1} + \varepsilon^m F_m(x, \tau, \varepsilon). \tag{4.16}$$

Moreover, because of (4.10), we have

$$U_\varepsilon^m(x, 0) = \bar{U}(x, \varepsilon) + O(\varepsilon^{m+1}). \tag{4.17}$$

In order to estimate F_m, we notice from (4.11) and (4.12) that

$$R(U_\varepsilon^m) = \sum_{k=m+1}^{\infty} \varepsilon^{k-1}\{P_{k\tau} - Q_U(P_0 + I_0)\, P_k - \mathscr{C}''(k, \underline{P+I})\}. \tag{4.18}$$

Here we are using that, for $k > m$, $I_k(x, \tau) \equiv 0$ and $P_k(x, \tau) = \sum_{h=k-m}^{k} (\tau^h/h!)(\partial^h U_{k-h}/\partial t^h)(x, 0)$ depends only on $U_0, U_1, ..., U_m$. Similarly, it follows from (4.13) that

$$\begin{aligned} R\left(\sum_{k=0}^{m} \varepsilon^k U_k(x, t)\right) &= R\left(\sum_{k=0}^{\infty} \varepsilon^k P_k(x, \tau)\right) \\ &= \sum_{k=m+1}^{\infty} \varepsilon^{k-1}\{P_{k\tau} - Q_U(P_0)\, P_k - \mathscr{C}''(k, \underline{P})\}. \end{aligned}$$

Thus, it follows from the definition of F_m, (4.15), and (4.18) that

$$\begin{aligned} F_m &= \varepsilon^{-m} R(U_\varepsilon^m) - \varepsilon^{-m} R\left(\sum_{k=0}^{m} \varepsilon^k U_k(x, t)\right) + O(\varepsilon) \\ &= O(\varepsilon) + \sum_{k=m+1}^{\infty} \varepsilon^{k-m-1}\{[Q_U(P_0) - Q_U(P_0 + I_0)]\, P_k \\ &\quad + \mathscr{C}''(k, \underline{P}) - \mathscr{C}''(k, \underline{P+I})\} \\ &= O(\varepsilon) - \sum_{k=m+1}^{\infty} \varepsilon^{k-m-1}[Q_{UU}(\cdots)\, I_0 P_k + \mathscr{C}''_U(k, \cdot)\, \underline{I}]. \end{aligned}$$

Here $\mathscr{C}''_U(k, \cdot)$ denotes the Fréchet derivative (with respect to the second argument) at an intermediate point. Useful estimates for F_m will be seen if each $I_k = I_k(x, \tau)$ exponentially decays to zero as τ goes to infinity.

5. ESTIMATES OF THE EXPANSIONS

In this section we show some regularity properties and estimates of U_ε^m, $Q_U(U_0)\, U_{m+1}$ and $F_m(x, \tau, \varepsilon)$ constructed in the previous section, see (4.16). For simplicity, we consider only the case where $m = 1$.

First of all, let us agree on some notations. For vectors $U, V \in \mathbf{C}^k$ and matrix $A \in \mathbf{C}^{l \times k}$ our basic inner product and norm are

$$\langle U, V \rangle = U^* V, \qquad |U| = \langle U, U \rangle^{1/2}, \quad \text{and} \quad |A| = \max\{|AU| : |U| = 1\}.$$

Let $\Omega=(0,1]^d$. L^2 is the space of square integrable (vector- or matrix-valued) functions on Ω. $\|A\|$ and (U, V) denote its norm and inner product, respectively. In case U, V and A are functions of another variable t as well as $x\in\Omega$, we write $\|A(t)\|$ and $(U(t), V(t))$ to remind the reader that the norm and the inner product are computed with respect to x while t is viewed as a parameter. Similar notations will be adopted for the function spaces introduced below. For a nonnegative integer s, the Sobolev space H^s is defined as the space of functions which and their distribution derivatives of order $\leqslant s$ are all in L^2. We use $\|A\|_s$ and $(U, V)_s$ to denote the norm and the inner product of H^s.

Let R be an open subset of some real Euclid space. We denote by $C(R)$ (resp. $C^k(R)$ with k being a positive integer or ∞) the space of continuous (resp. k-times continuously differentiable) functions on R. $C_b(R)$ (resp. $C_b^k(R)$) is a subspace of $C(R)$ (resp. $C^k(R)$) whose elements (resp. and all derivatives of order $\leqslant k$) are bounded on R. $C_b(R)$ and $C_b^k(R)$ are both Banach spaces respectively for the norms

$$|A|_{0,R}=\sup\{|A(x)|, x\in R\} \qquad \text{and} \qquad |A|_{k,R}=\sup\{|\partial^\alpha A|_{0,R}, |\alpha|\leqslant k\}.$$

We also adopt the multi-index notations in [10] and denote them by Greek alphabets α, β and so on. When it can be inferred from the context, the subscript R will frequently be omitted from the above notations. In addition, we denote by $C^k([0,T], \mathbf{X})$ the k-times continuously differentiable functions on $[0,T]$ with values in the Banach space $\mathbf{X}$.

We need the following well-known calculus inequalities in Sobolev spaces. In what follows, C_s denotes a generic constant depending only on s and d.

LEMMA 5.1. *Let s, s_1 and s_2 be three non-negative integers.*

(a) *If $s_3=\min\{s_1, s_2, s_1+s_2-s_0\}\geqslant 0$, then*

$$H^{s_1}H^{s_2}\subset H^{s_3}.$$

Here $s_0=[d/2]+1$ and the inclusion symbol $\subset$ implies the continuity of the embedding.

(b) *Suppose $s\geqslant s_0+1$, $A\in H^s$ and $U\in H^{s-1}$. Then for all multi-indices α with $|\alpha|\leqslant s$, $\partial^\alpha(AU)-A\partial^\alpha U\in L^2$ and*

$$\|\partial^\alpha(AU)-A\partial^\alpha U\|\leqslant C_s\,\|A\|_s\,\|U\|_{|\alpha|-1}.$$

(c) *Suppose $s\geqslant s_0$, $A\in C_b^s(G)$ and $V\in H^s(\Omega, G)$. Then $A(V(\cdot))\in H^s$ and*

$$\|A(V(\cdot))\|_s\leqslant C_s\,|A|_s\,(1+\|V\|_s^s).$$

The proof of this lemma can be found in [11, 22]. In fact, (a) is a slight modification of Lemma 2.1 in [11], while (b) and (c) follow directly from (a) (see [22]). See also Proposition 2.1 in [15].

Using the above lemma, we can easily prove (see [22])

LEMMA 5.2. *Let $s \geqslant s_0$ be an integer.*

(a) *Both $C([0, T], H^s)$ and $C([0, T], H^s) \cap C^1([0, T], H^{s-1})$ are algebra.*

(b) *Assume $F \in C^s(G)$ and $V \in C([0, T], H^s)$ with values in a compact subset of G. Then $F(V) \in C([0, T], H^s)$. Moreover, $F(V) \in C([0, T], H^s) \cap C^1([0, T], H^{s-1})$ if so is V and $F \in C^{s+1}(G)$.*

Our further arguments are based on the following assumption on the two solutions respectively to the limiting inner problem and the reduced problem.

Assumption. (1) The limiting inner problem (the equation for $\tilde{I}_0(x, \tau)$ in (4.14) with $\tilde{I}_0(x, 0) = \bar{U}(x, 0)$) has a unique solution $\tilde{I}_0(x, \tau)$ in $C([0, +\infty), H^{s+2})$, which decays exponentially to $\tilde{I}_0(x, +\infty) \in \mathscr{E}$ in H^{s+2} as τ goes to infinity. Here $s \geqslant s_0 + 1$.

(2) The reduced problem (4.5) and (4.7) for $k = 0$ with $U_0(x, 0) = \tilde{I}_0(x, +\infty)$ has a unique solution $U_0(x, t)$ in $C([0, T_0], H^{s+2}) \cap C^1([0, T_0], H^{s+1}) \cap C^2([0, T_0], H^s)$.

(3) $\{U_0(x, t) + \theta I_0(x, \tau) \colon (x, t, \tau, \theta) \in \Omega \times [0, T_0] \times [0, +\infty) \times [0, 1]\} \subset\subset G$.

We also assume that A_j ($j = 1, 2, ..., d$), Q, $\hat{S}$ and P are in $C^{s+2}(G)$ and $\bar{U}_1(x)$ in (4.10) belongs to H^s. Note that the assumption in (2) holds when $E(u)$ in (4.9) is smooth.

Let us first analyse U_1. On the basis of the above assumption, we use Lemma 5.2 to see that

$$A_j(U_0), \hat{S}(U_0), P(U_0) \in C([0, T_0], H^{s+1}) \cap C^1([0, T_0], H^s).$$

Thus, it follows from (4.8) with $k = 0$ and Lemma 5.2 that

$$P^{II}(U_0)\, U_1 \in C([0, T_0], H^{s+1}) \cap C^1([0, T_0], H^s). \tag{5.1}$$

On the other hand, recall from (4.7) with $k = 1$ that $w \equiv P^I(U_0)\, U_1$ satisfies a semilinear symmetrizable hyperbolic system

$$w_t + \sum_{j=1}^{d} a_j(x, t)\, w_{x_j} = f(w, x, t), \tag{5.2}$$

where each component of $f(w, x, t)$ is a polynomial of w. Using Lemma 5.2 and (5.1) we can easily show that the given functions in (5.2) satisfy the conditions of Theorem III.3 in [22]. Thus, we assume $w(x, 0) \in H^s$ for the moment and conclude from the theorem that there is a positive constant T_1 such that the above problem has a unique classical solution

$$w \equiv P^I(U_0)\, U_1 \in C([0, T_1], H^s) \cap C^1([0, T_1], H^{s-1}).$$

In view of (5.1), we have

$$U_1 \in C([0, T_1], H^s) \cap C^1([0, T_1], H^{s-1}). \tag{5.3}$$

Furthermore, recall from (4.6) that

$$Q_U(U_0)\, U_2 = U_{1t} + \sum_{j=1}^{d} A_j(U_0)\, U_{1x_j} - \mathscr{C}'(2, \underline{U}).$$

A direct calculation shows $\mathscr{C}'(2, \underline{U}) = (Q_{UU}(U_0)\, U_1)\, U_1/2 - \sum_{j=1}^d A_j(U_0)_{x_j} U_1$. Thus, we have

$$Q_U(U_0)\, U_2 \in C([0, T_1], H^{s-1}). \tag{5.4}$$

Next, we consider the equation in (4.14) for $\tilde{I}_1 = I_1 + U_1(x, 0)$ and choose $w(x, 0) \equiv P^I(U_0(x, 0))\, U_1(x, 0) \in H^s$. To this end, we set $V = P(U_0(x, 0))\, \tilde{I}_1$ and rewrite the equation as

$$\frac{dV}{d\tau} = S(x, \tau)\, V - S(x)\, P(U_0(x, 0))\, U_1(x, 0) + F(x, \tau), \tag{5.5}$$

where

$$\begin{aligned}
S(x, \tau) &= P(U_0(x, 0))\, Q_U(\tilde{I}_0)\, P^{-1}(U_0(x, 0)),\\
S(x) &= \mathrm{diag}(0, \hat{S}(U_0(x, 0))),\\
F(x, \tau) &\equiv P(U_0(x, 0))\, r_1(x, \tau)\\
&= \tau P(U_0(x, 0))[Q_U(\tilde{I}_0) - Q_U(U_0(x, 0))]\, U_{0t}(x, 0)\\
&\quad - P(U_0(x, 0)) \sum_j \{A_j(\tilde{I}_0)\, I_{0x_j}\\
&\quad + [A_j(\tilde{I}_0) - A_j(U_0(x, 0))]\, U_{0x_j}(x, 0)\}.
\end{aligned}$$

Since $\tilde{I}_0(x, \tau) \in C([0, \infty), H^{s+1})$, it follows from Lemma 5.2 that $Q_U(\tilde{I}_0(x, \tau))$ and thereby $S(x, \tau)$ is in $C([0, \infty), H^{s+1})$. Moreover, by using the assumption, we can also show $F(x, \tau) \in C([0, \infty), H^{s+1})$. Then, it is easy to see that the above problem has a unique solution in $C^1([0, +\infty), H^{s+1})$ if $V(x, 0) = P(U_0(x, 0))\, \bar{U}_1(x) \in H^{s+1}$. In view of (5.3), we have $I_1 \in C^1([0, +\infty), H^s)$. Thus, combining (5.3) and the assumption shows that for each $\varepsilon > 0$,

$$U_\varepsilon^1 \in C([0, T_1], H^s) \cap C^1([0, T_1], H^{s-1}). \tag{5.6}$$

Moreover, from (4.16) and (5.4) it follows that for each $\varepsilon > 0$,

$$F_1(\cdot, \cdot, \varepsilon) \in C([0, T_1], H^{s-1}). \tag{5.7}$$

Now we show that $\|I_1(\tau)\|_s$ decays exponentially to zero as τ goes to infinity. To do so, we first estimate $\|S(\cdot, \tau) - S(\cdot)\|_s$ for fixed τ. Since $Q_{UU} \in C^s(G)$ and $U_0(x, 0) + \theta I_0(x, \tau) \in C([0, 1], H^s)$, we have $Q_{UU}(U_0(x, 0) + \theta I_0(x, \tau)) \in C([0, 1], H^s)$. Thus, $\int_0^1 Q_{UU}(U_0(x, 0) + \theta I_0(x, \tau))\, d\theta \in H^s$ and from Lemma 5.1(c)

$$\left\| \int_0^1 Q_{UU}(U_0(x, 0) + \theta I_0(x, \tau))\, d\theta \right\|_s \leqslant C_s\, |Q_{UU}|_s\, (1 + \|U_0(0)\|_s^s + \|I_0(\tau)\|_s^s).$$

Because of the assumption, there exist positive constants μ and C such that $\|I_0(\tau)\|_s \leqslant Ce^{-\mu\tau}$. Thus, we obtain from Lemma 5.1(a)

$$\begin{aligned}
\|S(\cdot, \tau) - S(\cdot)\|_s &\equiv \|P(U_0(0))[Q_U(\tilde{I}_0(\tau)) - Q_U(U_0(0))]\, P^{-1}(U_0(0))\|_s \\
&\leqslant C_s\, \|P(U_0(0))\|_s\, \|P^{-1}(U_0(0))\|_s \\
&\quad \times \left\| \int_0^1 Q_{UU}(U_0(x, 0)\, 2, ..., d) + \theta I_0(x, \tau))\, d\theta \right\|_s \|I_0(\tau)\|_s \\
&\leqslant Ce^{-\mu\tau}.
\end{aligned} \tag{5.8}$$

Similarly, we can show that $\|F(\tau)\|_s$ decays exponentially to zero as τ goes to infinity.

With the above μ, we define $\tilde{V} = Ve^{-\mu\tau}$ and deduce from (5.5) that $\tilde{V}$ satisfies

$$\frac{d\tilde{V}}{d\tau} = [S(x, \tau) - \mu I]\, \tilde{V} - e^{-\mu\tau} S(x)\, P(U_0(0, x))\, U_1(x, 0) + e^{-\mu\tau} F(x, \tau). \tag{5.9}$$

Since $\hat{S}(x) \equiv \hat{S}(U_0(x, 0))$ is a stable matrix, there is a positive definite Hermitian matrix $\hat{E}(x)$ such that

$$\hat{E}(x)\, \hat{S}(x) + \hat{S}^*(x)\, \hat{E}(x) \leqslant -I_r. \tag{5.10}$$

Thus, with $E(x) = \operatorname{diag}(I_{n-r}, \hat{E}(x))$, we have

$$E(x)[S(x) - \mu I] + [S(x) - \mu I]^*\, E(x) \leqslant -\operatorname{diag}(2\mu I_{n-r}, I_r + 2\mu \hat{E}(x)).$$

On the other hand, note that $|S(\cdot, \tau) - S(\cdot)|_0$ tends to zero as τ goes to infinity. Then for any $\eta > 0$, there is $\tau_0 > 0$ such that

$$E(x)[S(x, \tau) - S(x)] + [S(x, \tau) - S(x)]^* E(x) \leqslant \eta I$$

for $\tau \geqslant \tau_0$ and all x. Thus, we have

$$E(x)[S(x, \tau) - \mu I] + [S(x, \tau) - \mu I]^* E(x) \leqslant -I \tag{5.11}$$

if η is small enough.

At this point we establish the following

Lemma 5.3. *Suppose $f(x, \tau) \in C([0, \infty), L^2)$, $\|f(\tau)\|$ decays exponentially to zero as τ goes to infinity, and $E(x) \in L^\infty$ is a uniformly positive definite Hermitian matrix such that for all sufficiently large τ and all x,*

$$E(x)\, A(x, \tau) + A^*(x, \tau)\, E(x) \leqslant -I.$$

If $V(x, \tau) \in C([0, \infty), L^2)$ satisfies

$$\frac{dV}{d\tau} = A(x, \tau)\, V + f(x, \tau), \tag{5.12}$$

then $\|V(\tau)\|$ decays exponentially to zero as τ goes to infinity. Moreover, if $V(x, \tau), f(x, \tau) \in C([0, \infty), H^s)$ and $\|f(\tau)\|_s$ decays exponentially to zero as τ goes to infinity, then $\|V(\tau)\|_s$ decays exponentially to zero as τ goes to infinity.

Proof. According to the conditions, there exist positive constants μ, τ_0 such that

$$E(x)\, A(x, \tau) + A^*(x, \tau)\, E(x) \leqslant -I, \qquad \|f(\tau)\|^2 \leqslant \tau_0 e^{-\mu\tau}, \quad |E|_0 \leqslant \tau_0 I$$

for $\tau \geqslant \tau_0$ and all x. Multiplying the equation in (5.12) with $V^*(x, \tau)\, E(x)$, taking the real parts and integrating over Ω, we obtain, for $\tau \geqslant \tau_0$,

$$\begin{aligned}\frac{d}{d\tau}(V(\tau), EV(\tau)) &\leqslant -(V(\tau), V(\tau)) + 2\,\mathrm{Re}(V(\tau), Ef(\tau)) \\ &\leqslant -\frac{(V(\tau), V(\tau))}{2} + 2\,|E|_0^2\, \|f(\tau)\|^2 \\ &\leqslant -\frac{1}{2\tau_0}(V(\tau), EV(\tau)) + 2\tau_0^2 \tau_0 e^{-\mu\tau}.\end{aligned}$$

Thus,

$$(V(\tau), EV(\tau)) \leqslant \exp\left(\frac{-\tau}{2\tau_0}\right) \sqrt{e}\,(V(\tau_0), EV(\tau_0))$$
$$+ 2\tau_0^3 \exp\left(\frac{-\tau}{2\tau_0}\right) \int_{\tau_0}^{\tau} \exp\left(\frac{\sigma}{2\tau_0} - \mu\sigma\right) d\sigma.$$

This and the uniform positiveness of $E(x)$ together imply the exponential decay of $\| V(\tau)\|$ to zero.

For the other statement, note that for all multi-indices α such that $|\alpha| \leqslant s$,

$$\|A(\tau)\, V_\alpha(\tau) - (A(\tau)\, V(\tau))_\alpha\| \leqslant C_s\, \|A(\tau)\|_s\, \|V(\tau)\|_{|\alpha|-1}$$

due to Lemma 5.1(b). Therefore, we may use induction on $|\alpha|$ to prove the exponential decay of $\| V(\tau)\|_s$ to zero. This completes the proof. ∎

Because of the inequality in (5.11), we apply Lemma 5.3 to the equation in (5.9) to see that $\|\tilde{V}(\tau)\|_s$ decays exponentially to zero as τ goes to infinity. Set $W = V - P(U_0(x, 0))\, U_1(x, 0)$ and rewrite the equation in (5.5) as

$$\frac{dW(x, \tau)}{d\tau} = S(x)\, W(x, \tau) + \underbrace{[S(x, \tau) - S(x)]\, V(x, \tau) + F(x, \tau)}_{f(x, \tau)}.$$

Since the first $(n-r)$ rows of $S(x)$ vanishes and $\|f(\tau)\|_s$ decays exponentially to zero as τ goes to infinity, $\|dV^I(\tau)/d\tau\|_s$ decays exponentially to zero as τ goes to infinity. Here we have used the superscript I (II) to denote the first $(n-r)$ (last r) components of a vector. Therefore, it is clear that $V^I_\infty(x) \equiv \lim_{\tau \to \infty} V^I(x, \tau)$ exists in H^s. Thus, we take $P^I(U_0(x, 0))\, U_1(x, 0) = V^I_\infty(x)$. Moreover,

$$\frac{dW^{II}(x, \tau)}{d\tau} = \hat{S}(x)\, W^{II}(x, \tau) + f^{II}(x, \tau).$$

In view of (5.10), we apply Lemma 5.3 to the last equation to see that $\| W^{II}(\tau)\|_s$ decays exponentially to zero as τ goes to infinity. Consequently, we arrive at

$$\|I_1(\tau)\|_s \equiv \|P^{-1}(U_0(0))\, W(\tau)\|_s \leqslant Ce^{-\mu\tau}. \tag{5.13}$$

Having (5.3), (5.6) and the estimate on I_1, we write out the expressions of F_1. Then as in obtaining (5.8), we get

$$\|F_1(\tau, \varepsilon)\|_{s-1} \leqslant C\varepsilon + Ce^{-\mu\tau} \tag{5.14}$$

for sufficiently small ε.

In conclusion, we have estimates in (5.6) for U_ε^1, (5.4) for $Q_U(U_0)\, U_2$ and (5.7) + (5.14) for $F_1(x, \tau, \varepsilon)$.

6. VALIDITY OF EXPANSIONS AND EXISTENCE

Having constructed a sufficiently regular asymptotic approximation U_ε^m (with $m=1$ for simplicity) for the initial value problem in (1.1), we prove here the existence of a smooth solution U^ε in the ε-independent time interval where U_ε^1 is well defined, and the validity of the approximation under the stability condition and similar regularity assumptions as that in the previous section.

For the sake of exactness, we refer to the results in the previous section and make the following assumption for U_ε^1, $Q_U(U_0)\, U_2$ and F_1 in (4.16) throughout this section. To this end, let s be an integer such that $s \geqslant s_0 + 1$ and denote by $\mathscr{R}(V)$ the range of $V = V(x, t, \varepsilon)$ for $(x, t, \varepsilon) \in \Omega \times [0, T_1] \times [0, \varepsilon_0]$. Here ε_0 is a small positive constant and T_1 is such that U_ε^1 is well defined in $[0, T_1]$ (see (5.3)).

Assumption A. (1) A_j $(j = 0, 1, 2, ..., d)$, $Q \in C^\infty(G)$ and $\bar{U}(\cdot, \varepsilon) \in H^s$ is periodic in x with period $(1, 1, ..., 1) \in \mathbf{R}^d$.

(2) The reduced problem has a unique solution $U_0 \in C([0, T_1], H^s)$ such that $P(U_0) \in C([0, T_1], H^{s+1}) \cap C^1([0, T_1], H^s)$ and $P^{-1}(U_0)$, $Q_U(U_0)\, U_2 \in C([0, T_1], H^s)$.

(3) $U_\varepsilon^1 \in C([0, T_1], H^{s+1})$ for each ε, $\|\bar{U}(\cdot, \varepsilon) - U_\varepsilon^1(0)\|_s = O(\varepsilon^{3/2})$ and $\|U_\varepsilon^1(t)\|_{s+1}$ is uniformly bounded with respect to ε and t.

(4) There is a convex open set G_0 such that $\mathscr{R}(U_0) \cup \mathscr{R}(U_\varepsilon^1) \subset G_0 \subset\subset G$.

(5) $F_1(\cdot, \cdot, \varepsilon) \in C([0, T_1], H^s)$ for each ε and $\|F_1(\tau, \varepsilon)\|_s \leqslant C\varepsilon + Ce^{-\mu\tau}$ with μ being a positive constant.

From the construction of U_ε^1, we can see that there is a *boundary-layer* function $B_\varepsilon(t)$, a function satisfying

$$\sup_\varepsilon \int_0^{T_1} B_\varepsilon(t)\, dt < \infty,$$

such that

$$\frac{\| U_\varepsilon^1(t) - U_0(t) \|_s}{\varepsilon} \leqslant B_\varepsilon(t). \tag{6.1}$$

On the other hand, because of the well-known embedding inequality

$$|\cdot|_0 \leqslant C_{s_0} \|\cdot\|_{s_0}, \tag{6.2}$$

U_0 and U_ε^1 are continuously differentiable due to the above assumption and the corresponding equations.

Fix $\varepsilon > 0$ and assume $\mathscr{R}(\bar{U}) \subset\subset G_0$. Then, according to Theorem 2.1 in [15], there exists $T_\varepsilon > 0$ such that the initial value problem in (1.1) for the symmetrizable hyperbolic system has a unique classical solution U^ε satisfying $U^\varepsilon(x, t) \in \bar{G}_0$ for $(x, t) \in \Omega \times [0, T_\varepsilon]$ and

$$U^\varepsilon \in C([0, T_\varepsilon], H^s) \cap C^1([0, T_\varepsilon], H^{s-1}).$$

Namely,

$$\begin{gathered} U_t^\varepsilon + \sum_{j=1}^d A_j(U^\varepsilon)\, U_{x_j}^\varepsilon = \frac{Q(U^\varepsilon)}{\varepsilon}, \\ U(x, 0) = \bar{U}(x, \varepsilon). \end{gathered} \tag{6.3}$$

Without loss of generality, we assume that $[0, T_\varepsilon]$ is the *maximal time interval* where the H^s-solution U^ε exists and $U^\varepsilon(x, t) \in \bar{G}_0$ for $(x, t) \in \Omega \times [0, T_\varepsilon]$. Note that T_ε may tend to zero as so does ε.

In order to show $T_\varepsilon \geqslant T_1$, we first prove

THEOREM 6.1. *Under the stability condition and the assumption A, there exists a constant K, depending only on the assumption, such that*

$$\| U^\varepsilon(t) - U_\varepsilon^1(t) \|_s \leqslant K\varepsilon^{3/2}$$

for ε sufficiently small and $t \in [0, \min\{T_\varepsilon, T_1\}]$.

Before proving this theorem, we show that one of its consequences is $T_\varepsilon \geqslant T_1$ if $\mathscr{R}(U_\varepsilon^1) \subset\subset G_0$. In fact, if $T_\varepsilon < T_1$, then Theorem 6.1 gives

$$\| U^\varepsilon(T_\varepsilon) - U_\varepsilon^1(T_\varepsilon) \|_s \leqslant K\varepsilon^{3/2}.$$

Thus, it follows from the embedding inequality in (6.2) and $\mathscr{R}(U_\varepsilon^1(T_\varepsilon)) \subset\subset G_0$ that $\mathscr{R}(U^\varepsilon(T_\varepsilon)) \subset\subset G_0$ if ε is small enough. Now we could apply Theorem 2.1 in [15], beginning at the time T_ε, to continue this solution beyond T_ε. This is a contradiction. Consequently, we have proved

THEOREM 6.2. *Under the stability condition and the assumption A, assume $\mathscr{R}(U^1_\varepsilon) \subset\subset G_0$. Then the initial value problem in* (1.1) *has a unique classical solution U^ε, defined in $\Omega \times [0, T_1]$, satisfying $U^\varepsilon(x, t) \in G_0$ for $(x, t) \in \Omega \times [0, T_1]$ and*

$$U^\varepsilon \in C([0, T_1], H^s) \cap C^1([0, T_1], H^{s-1}).$$

Another consequence of Theorem 6.1 is that we can write

$$U^\varepsilon(x, t) = U_0(x, t) + I_0(x, t/\varepsilon) + \varepsilon U_1(x, t) + \varepsilon I_1(x, t/\varepsilon) + O(\varepsilon^{3/2}) \tag{6.4}$$

due to the form of U^1_ε. This is just the asymptotic expansion of $U^\varepsilon(x, t)$.

Now we turn to prove Theorem 6.1 and begin with the following nonlinear Gronwall-type inequality.

LEMMA 6.3. *Suppose $\psi(t)$ is a positive C^1-function of $t \in [0, T)$ with $T \leqslant \infty$, $m > 1$ and $b_1(t)$, $b_2(t)$ are integrable on $[0, T)$. If*

$$\psi'(t) \leqslant b_2(t)\, \psi^m(t) + b_1(t)\, \psi(t),$$

then there exists $\delta > 0$, depending only on m, C_{1b}, and C_{2b}, such that

$$\sup_{t \in [0, T)} \psi(t) \leqslant e^{C_{1b}},$$

whenever $\psi(0) \in (0, \delta]$. Here

$$C_{1b} = \sup_{t \in [0, T)} \int_0^t b_1(t')\, dt' \qquad \text{and} \qquad C_{2b} = \int_0^T \max\{b_2(t), 0\}\, dt.$$

Proof. Set $\Phi(t) = \psi^{1-m}(t)$ and compute that $\Phi(t)$ satisfies

$$\Phi'(t) \geqslant (1-m)(b_2(t) + b_1(t)\, \Phi(t)) \geqslant (1-m)(\max\{b_2(t), 0\} + b_1(t)\, \Phi(t)).$$

Therefore,

$$\Phi(t) \exp\left((m-1) \int_0^t b_1(t')\, dt'\right) - \Phi(0)$$
$$\geqslant (1-m) \int_0^t \max\{b_2(t'), 0\}\, dt' \exp\left((m-1) \int_0^{t'} b_1(t'')\, dt''\right).$$

Since

$$\int_0^t b_1(t')\,dt' \leqslant C_{1b} \qquad \text{and} \qquad \int_0^t \max\{b_2(t'), 0\}\,dt' \leqslant C_{2b},$$

we deduce that

$$e^{(m-1)\,C_{1b}}\Phi(t) \geqslant \psi^{1-m}(0) - (m-1)\,C_{2b}e^{(m-1)\,C_{1b}}. \tag{6.5}$$

By choosing δ so small that $\delta^{1-m} - (m-1)\,C_{2b}e^{(m-1)\,C_{1b}} \geqslant 1$, (6.5) directly leads to the conclusion of the lemma and the proof is complete. ∎

The Proof of Theorem 6.1.

Recall the equation in (4.16) and set $\tilde{W} = U_\varepsilon^1 - U^\varepsilon$. Then we deduce from (6.3) and (4.16) that

$$\begin{aligned}\tilde{W}_t + \sum_{j=1}^d A_j(U^\varepsilon)\,\tilde{W}_{x_j} = {} & \frac{Q(U_\varepsilon^1) - Q(U^\varepsilon)}{\varepsilon} + \varepsilon Q_U(U_0)\,U_2 + \varepsilon F_1\\ & + \sum_{j=1}^d [A_j(U^\varepsilon) - A_j(U_\varepsilon^1)]\,U_{\varepsilon x_j}^1,\\ \tilde{W}(x, 0) = {} & U_\varepsilon^1(x, 0) - \bar{U}(x, \varepsilon).\end{aligned}$$

Define $W = P(U_0)\,\tilde{W} \equiv P\tilde{W}$. Then W satisfies

$$\begin{aligned}W_t + \sum_{j=1}^d PA_j(U^\varepsilon)\,P^{-1}W_{x_j} = {} & \frac{P[Q(U_\varepsilon^1) - Q(U^\varepsilon)]}{\varepsilon} + \varepsilon PQ_U(U_0)\,U_2 + \varepsilon PF_1\\ & + P\sum_{j=1}^d [A_j(U^\varepsilon) - A_j(U_\varepsilon^1)]\,U_{\varepsilon x_j}\\ & + \left[P_t + \sum_{j=1}^d PA_j(U^\varepsilon)\,P^{-1}P_{x_j}\right]P^{-1}W,\\ W(x, 0) = {} & P[U_\varepsilon^1(x, 0) - \bar{U}(x, \varepsilon)].\end{aligned}$$

Differentiating the last equation with ∂^α for $|\alpha| \leqslant s$ and setting $W_\alpha = \partial^\alpha W$, we get

$$\begin{aligned}W_{\alpha t} + \sum_{j=1}^d PA_j(U^\varepsilon)\,P^{-1}W_{\alpha x_j} = {} & \frac{PQ_U(U_0)\,P^{-1}}{\varepsilon}\,W_\alpha + F_1^\alpha + F_2^\alpha,\\ W_\alpha(x, 0) = {} & \{P[U_\varepsilon^1(x, 0) - \bar{U}(x, \varepsilon)]\}_\alpha.\end{aligned} \tag{6.6}$$

Here

$$F_1^\alpha = \varepsilon(PQ_U(U_0)\, U_2)_\alpha + \frac{[PQ_U(U_0)\, P^{-1}W]_\alpha - PQ_U(U_0)\, P^{-1}W_\alpha}{\varepsilon},$$

$$\begin{aligned} F_2^\alpha = {} & \varepsilon(PF_1)_\alpha + \frac{\{P[Q(U_\varepsilon^1) - Q(U^\varepsilon)] - PQ_U(U_0)\, P^{-1}W\}_\alpha}{\varepsilon} \\ & + \left\{P \sum_{j=1}^d [A_j(U^\varepsilon) - A_j(U_\varepsilon^1)]\, U^1_{\varepsilon x_j}\right\}_\alpha \\ & + \left\{[P_t + \sum_{j=1}^d PA_j(U^\varepsilon)\, P^{-1}P_{x_j}]\, P^{-1}W\right\}_\alpha \\ & + \sum_{j=1}^d \{PA_j(U^\varepsilon)\, P^{-1}W_{\alpha x_j} - [PA_j(U^\varepsilon)\, P^{-1}W_{x_j}]_\alpha\} \\ \equiv {} & f_1^\alpha + f_2^\alpha + f_3^\alpha + f_4^\alpha + f_5^\alpha. \end{aligned}$$

To be easy of understanding, we group the following arguments into lemmas.

LEMMA 6.4. *Under the stability condition and the assumption A, we have*

$$\begin{aligned} \frac{d}{dt} \int_\Omega e(W_\alpha(x, t))\, dx + \frac{\|W_\alpha^{II}(t)\|^2}{2\varepsilon} \leqslant {} & C\varepsilon^3 + \frac{C\, \|W^{II}(t)\|^2_{|\alpha|-1}}{\varepsilon} + C\, \|W_\alpha(t)\|\, \|F_2^\alpha\| \\ & + C \int_\Omega \left\{1 + |U_t^\varepsilon(x, t)| + \sum_{j=1}^d |U_{x_j}^\varepsilon(x, t)| \right. \\ & \left. + \frac{|U^\varepsilon(x, t) - U_0(x, t)|}{\varepsilon}\right\} |W_\alpha(x, t)|^2\, dx. \end{aligned}$$

Here $e(W_\alpha) = W_\alpha^* P^{-*} A_0(U^\varepsilon)\, P^{-1} W_\alpha$, $\|\cdot\|_{-1} = 0$ and C is a generic constant depending only on the assumption A.

Proof. Recall that $A_0(U^\varepsilon)$ and $A_0(U^\varepsilon)\, A_j(U^\varepsilon)$ $(j = 1, 2, ..., d)$ are all symmetric. Multiplying the equation in (6.6) with $W_\alpha^* P^{-*} A_0(U^\varepsilon)\, P^{-1}$ from the left, we get

$$
\begin{aligned}
&e(W_\alpha)_t+\sum_{j=1}^{d}\{W_\alpha^* P^{-*}A_0(U^\varepsilon)\,A_j(U^\varepsilon)\,P^{-1}W_\alpha\}_{x_j}\\
&\quad=\frac{2}{\varepsilon}\operatorname{Re} W_\alpha^* P^{-*}A_0(U_0)\,Q_U(U_0)\,P^{-1}W_\alpha+2\operatorname{Re} W_\alpha^* P^{-*}A_0(U^\varepsilon)\,P^{-1}F_1^\alpha\\
&\qquad+\frac{2}{\varepsilon}\operatorname{Re} W_\alpha^* P^{-*}[A_0(U^\varepsilon)-A_0(U_0)]\,Q_U(U_0)\,P^{-1}W_\alpha\\
&\qquad+2\operatorname{Re} W_\alpha^* P^{-*}A_0(U^\varepsilon)\,P^{-1}F_2^\alpha\\
&\qquad+W_\alpha^*\left\{\frac{\partial[P^{-*}A_0(U^\varepsilon)\,P^{-1}]}{\partial t}+\sum_{j=1}^{d}\frac{\partial[P^{-*}A_0(U^\varepsilon)\,A_j(U^\varepsilon)\,P^{-1}]}{\partial x_j}\right\}W_\alpha.
\end{aligned}
\tag{6.7}
$$

Now let us analyse the right-hand side of (6.7) term-by-term,

$$
\begin{aligned}
&\frac{\partial[P^{-*}A_0(U^\varepsilon)\,P^{-1}]}{\partial t}+\sum_{j=1}^{d}\frac{\partial[P^{-*}A_0(U^\varepsilon)\,A_j(U^\varepsilon)\,P^{-1}]}{\partial x_j}\\
&\qquad\leqslant C\left(1+|U_t^\varepsilon|+\sum_{j=1}^{d}|U_{x_j}^\varepsilon|\right),
\end{aligned}
\tag{6.8}
$$

$$2\operatorname{Re} W_\alpha^* P^{-*}A_0(U^\varepsilon)\,P^{-1}F_2^\alpha\leqslant C\,|W_\alpha|\,|F_2^\alpha|, \tag{6.9}$$

$$2\operatorname{Re} W_\alpha^* P^{-*}[A_0(U^\varepsilon)-A_0(U_0)]\,Q_U(U_0)\,P^{-1}W_\alpha\leqslant C\,|U^\varepsilon-U_0|\,|W_\alpha|^2. \tag{6.10}$$

Since

$$A_0(U_0)\,Q_U(U_0)+Q_U^*(U_0)\,A_0(U_0)\leqslant -P^*\begin{pmatrix}0&0\\0&I_r\end{pmatrix}P,$$

we have

$$2\operatorname{Re} W_\alpha^* P^{-*}A_0(U_0)\,Q_U(U_0)\,P^{-1}W_\alpha\leqslant -|W_\alpha^{II}|^2. \tag{6.11}$$

In addition, thanks to (i) in the stability condition,

$$PQ_U(U_0)\,P^{-1}=\begin{pmatrix}0&0\\ &\hat{S}(U_0)\end{pmatrix},$$

the first $(n-r)$ components F_1^α vanish. Moreover, $P^{-*}A_0(U_0)\,P^{-1}$ is of the block-diagonal form (Theorem 2.2). Thus, we have

$$\begin{aligned}
&2 \operatorname{Re} W_\alpha^* P^{-*} A_0(U^\varepsilon)\, P^{-1} F_1^\alpha \\
&\quad = 2 \operatorname{Re} W_\alpha^* P^{-*} A_0(U_0)\, P^{-1} F_1^\alpha \\
&\qquad + 2 \operatorname{Re} W_\alpha^* P^{-*} [A_0(U^\varepsilon) - A_0(U_0)]\, P^{-1} F_1^\alpha \\
&\quad \leqslant \frac{|W_\alpha^{II}|^2}{2\varepsilon} + C\varepsilon\, |F_1^\alpha|^2 + C\, |W_\alpha|\, |U^\varepsilon - U_0|\, |F_1^\alpha| \\
&\quad \leqslant \frac{|W_\alpha^{II}|^2}{2\varepsilon} + C\varepsilon\, |F_1^\alpha|^2 + \frac{C\, |U^\varepsilon - U_0|^2}{\varepsilon} |W_\alpha|^2 \\
&\quad \leqslant \frac{|W_\alpha^{II}|^2}{2\varepsilon} + C\varepsilon\, |F_1^\alpha|^2 + \frac{C\, |U^\varepsilon - U_0|}{\varepsilon}\, |W_\alpha|^2
\end{aligned} \tag{6.12}$$

and

$$\begin{aligned}
\|F_1^\alpha\| &\leqslant \varepsilon\, \|(PQ_U(U_0)\, U_2)_\alpha(t)\| + \frac{1}{\varepsilon}\, \|[PQ_U(U_0)\, P^{-1} W]_\alpha \\
&\qquad - PQ_U(U_0)\, P^{-1} W_\alpha\| \\
&\leqslant C\varepsilon + \frac{C\, \|W^{II}(t)\|_{|\alpha|-1}}{\varepsilon}.
\end{aligned} \tag{6.13}$$

Here we have used Lemma 5.1 and the assumption A.

Having the estimates in (6.8)–(6.13) and using the periodicity, we integrate the two sides of the relation in (6.7) over $x \in \Omega \equiv (0, 1]^d$ to conclude the lemma.

LEMMA 6.5. *Let*

$$\Delta(t) = \frac{\|U_\varepsilon^1(t) - U^\varepsilon(t)\|_s}{\varepsilon}.$$

Then

$$\begin{aligned}
&|U_t^\varepsilon(t)|_0 + \sum_{j=1}^d |U_{x_j}^\varepsilon(t)|_0 + \frac{|U^\varepsilon(t) - U_0(t)|_0}{\varepsilon} \leqslant C + CB_\varepsilon(t) + C\Delta(t), \\
&\|F_2^\alpha\| \leqslant C\varepsilon^2 + C\varepsilon e^{-\mu\tau} + C(1 + B_\varepsilon(t))(1 + \Delta^{s+1}(t))\, \|W\|_{|\alpha|}.
\end{aligned} \tag{6.14}$$

Proof. It follows from the equation for U^ε and $Q(U_0) = 0$ that

$$|U_t^\varepsilon| \leqslant \sum_{j=1}^d |A_j(U^\varepsilon)\, U_{x_j}^\varepsilon| + \frac{|Q(U^\varepsilon) - Q(U_0)|}{\varepsilon} \leqslant C \sum_{j=1}^d |U_{x_j}^\varepsilon| + \frac{C\, |U^\varepsilon - U_0|}{\varepsilon}.$$

On the other hand, because $s \geqslant s_0+1$, we use the embedding inequality in (6.2) to obtain

$$
\begin{aligned}
|U^\varepsilon_{x_j}(x,t)| &\leqslant C_{s_0}\,\|U^\varepsilon(t)\|_s \leqslant C_{s_0}\,\|U^\varepsilon(t)-U^1_\varepsilon(t)\|_s + C_{s_0}\,\|U^1_\varepsilon(t)\|_s\\
&\leqslant C_{s_0}\varepsilon\Delta(t)+C,\\
|U^\varepsilon(x,t)-U_0(x,t)| &\leqslant C_{s_0}\,\|U^\varepsilon(t)-U_0(t)\|_s\\
&\leqslant C_{s_0}\,\|U^\varepsilon(t)-U^1_\varepsilon(t)\|_s + C_{s_0}\,\|U^1_\varepsilon(t)-U_0(t)\|_s\\
&\leqslant C_{s_0}\varepsilon\Delta(t)+C_{s_0}\varepsilon B_\varepsilon(t).
\end{aligned}
$$

Thus we have the first line in (6.14).

For $\|F^\alpha_2\|$, recall that $F^\alpha_2=\sum_{k=1}^5 f^\alpha_k$. Since $s \geqslant \max\{|\alpha|, s_0+1\}$, we use Lemma 5.1.a with $s_1=|\alpha|$ and $s_2=s$ and the assumption on $\|F_1\|_s$ to obtain

$$
\|f^\alpha_1\| \equiv \varepsilon\,\|(PF_1)_\alpha\| \leqslant \varepsilon\,\|(PF_1)\|_{|\alpha|} \leqslant C_s\varepsilon\,\|P\|_s\,\|F_1\|_{|\alpha|} \leqslant C\varepsilon^2 + C\varepsilon e^{-\mu\tau}. \tag{6.15}
$$

Furthermore, by using Lemma 5.1 we have

$$
\begin{aligned}
\|f^\alpha_5\| &\equiv \left\| \sum_{j=1}^d \{PA_j(U^\varepsilon)\,P^{-1}W_{\alpha x_j} - [PA_j(U^\varepsilon)\,P^{-1}W_{x_j}]_\alpha\} \right\|\\
&\leqslant C_s \sum_{j=1}^d \|PA_j(U^\varepsilon)\,P^{-1}\|_s\,\|W_{x_j}\|_{|\alpha|-1}\\
&\leqslant C_s \sum_{j=1}^d \|P\|_s\,\|A_j(U^\varepsilon)\|_s\,\|P^{-1}\|_s\,\|W\|_{|\alpha|}\\
&\leqslant C \sum_{j=1}^d |A_j|_s\,(1+\|U^\varepsilon\|^s_s)\,\|W\|_{|\alpha|}\\
&\leqslant C(1+2^s\,\|U^1_\varepsilon\|^s_s + 2^s\,\|U^\varepsilon - U^1_\varepsilon\|^s_s)\,\|W\|_{|\alpha|}\\
&\leqslant C(1+\Delta^s(t))\,\|W\|_{|\alpha|}.
\end{aligned}
\tag{6.16}
$$

For the last two inequalities, we have used the elementary inequality $(a+b)^s \leqslant 2^s(a^s+b^s)$ for $a, b \geqslant 0$ and that $\|U^1_\varepsilon\|_s$ is uniformly bounded with respect to ε. Moreover,

$$
\|f^\alpha_4\| \equiv \left\| \left\{ \left[P_t + \sum_{j=1}^d PA_j(U^\varepsilon)\,P^{-1}P_{x_j} \right] P^{-1}W \right\}_\alpha \right\| \leqslant C(1+\Delta^s(t))\,\|W\|_{|\alpha|}. \tag{6.17}
$$

Note that $\sup_{\varepsilon, t} \|U_\varepsilon^1(t)\|_{s+1} < \infty$ and

$$A_j(U^\varepsilon) - A_j(U_\varepsilon^1) = -\int_0^1 A_{jU}(U_\varepsilon^1 + \theta(U^\varepsilon - U_\varepsilon^1))\, d\theta P^{-1} W.$$

Then

$$\|f_3^\alpha\| \equiv \left\| \left\{ P \sum_{j=1}^d \left[A_j(U^\varepsilon) - A_j(U_\varepsilon^1)\right] U_{\varepsilon x_j}^1 \right\}_\alpha \right\| \leqslant C(1 + \Delta^s(t))\, \|W\|_{|\alpha|}. \quad (6.18)$$

To estimate f_2^α, set $U(\theta) = U_\varepsilon^1 - U_0 + (1-\theta)(U^\varepsilon - U_\varepsilon^1)$. Then

$$\|U(\theta)\|_s \leqslant \|U_\varepsilon^1 - U_0\|_s + \|U^\varepsilon - U_\varepsilon^1\|_s \leqslant \varepsilon(B_\varepsilon(t) + \Delta(t)),$$

$$P[Q(U_\varepsilon^1) - Q(U^\varepsilon)] - PQ_U(U_0)\, P^{-1} W$$

$$= P \int_0^1 \int_0^1 U(\theta)\, Q_{UU}(U_0 + \tau U(\theta))\, d\tau\, d\theta P^{-1} W$$

and therefore

$$\|f_2^\alpha\| \equiv \frac{\|\{P[Q(U_\varepsilon^1) - Q(U^\varepsilon)] - PQ_U(U_0)\, P^{-1} W\}_\alpha\|}{\varepsilon}$$

$$\leqslant \frac{C}{\varepsilon} \int_0^1 \int_0^1 \|U(\theta)\|_s\, \|Q_{UU}(U_0 + \tau U(\theta))\|_s\, d\theta\, d\tau\, \|W\|_{|\alpha|}$$

$$\leqslant C(B_\varepsilon(t) + \Delta(t))(1 + \Delta^s(t))\, \|W\|_{|\alpha|}. \quad (6.19)$$

Combining the estimates in (6.15)–(6.19) and using the elementary inequality $a^k \leqslant 1 + a^{s+1}$ for $a \geqslant 0$ and $0 \leqslant k \leqslant s+1$ yields the second line in (6.14). ∎

Substituting the estimates in (6.14) into the inequality in Lemma 6.4, we get

$$\frac{d}{dt} \int_\Omega e(W_\alpha(x, t))\, dx + \frac{\|W_\alpha^{II}(t)\|^2}{2\varepsilon}$$

$$\leqslant C\varepsilon^3 + C\varepsilon^2 e^{-2\mu\tau} + \frac{C\, \|W^{II}(t)\|_{|\alpha|-1}^2}{\varepsilon}$$

$$+ C(1 + B_\varepsilon(t))(1 + \Delta^{s+1}(t))\, \|W\|_{|\alpha|}^2.$$

Note that $C^{-1} |W_\alpha|^2 \leqslant e(W_\alpha) \leqslant C\, |W_\alpha|^2$, $\|W_\alpha(0)\|^2 \leqslant C\, \|\bar{U}(\cdot, \varepsilon) - U_\varepsilon^1(0)\|_{|\alpha|}^2 \leqslant C\varepsilon^3$ and $\int_0^T e^{-2\mu\tau}\, dt \leqslant \varepsilon/2\mu$. Integrating the last inequality for $t \in [0, T] \subset [0, \min\{T_\varepsilon, T_1\}]$ gives

$$\|W_\alpha(T)\|^2 + \frac{1}{\varepsilon}\int_0^T \|W_\alpha{}^{II}(t)\|^2\,dt$$

$$\leqslant C(1+T)\,\varepsilon^3 + \frac{C}{\varepsilon}\int_0^T \|W^{II}(t)\|^2_{|\alpha|-1}\,dt$$

$$+C\int_0^T (1+B_\varepsilon(t))(1+\Delta^{s+1}(t))\,\|W(t)\|^2_{|\alpha|}\,dt.$$

Let k be an integer such that $0 \leqslant k \leqslant s$. We sum up the last inequality for all α with $0 \leqslant |\alpha| \leqslant k$ to obtain

$$\|W(T)\|^2_k + \frac{1}{\varepsilon}\int_0^T \|W^{II}(t)\|^2_k\,dt$$

$$\leqslant C(1+T)\,\varepsilon^3 + \frac{C}{\varepsilon}\int_0^T \|W^{II}(t)\|^2_{k-1}\,dt$$

$$+C\int_0^T (1+B_\varepsilon(t))(1+\Delta^{s+1}(t))\,\|W(t)\|^2_k\,dt. \tag{6.20}$$

A simple iteration based on this inequality and $\|\cdot\|_{-1}=0$ leads to

$$\frac{1}{\varepsilon}\int_0^T \|W^{II}(t)\|^2_k\,dt \leqslant C(1+T)\,\varepsilon^3 + C\int_0^T (1+B_\varepsilon(t))(1+\Delta^{s+1}(t))\,\|W(t)\|^2_k\,dt. \tag{6.21}$$

Thus, it follows from (6.20) with $k=s$ that

$$\|W(T)\|^2_s \leqslant C(1+T_1)\,\varepsilon^3 + C\int_0^T (1+B_\varepsilon(t))(1+\Delta^{s+1}(t))\,\|W(t)\|^2_s\,dt.$$

Applying Gronwall's lemma to the last inequality leads to

$$\|W(T)\|^2_s \leqslant C\varepsilon^3(1+T_1)\exp\left(C\int_0^T (1+B_\varepsilon(t))(1+\Delta^{s+1}(t))\,dt\right). \tag{6.22}$$

Denote by $\varepsilon^2\phi(T)$ the right-hand side of (6.22), that is,

$$\phi(T) = C(T_1+1)\,\varepsilon\exp\left(C\int_0^T (1+B_\varepsilon(t))(1+\Delta^{s+1}(t))\,dt\right).$$

Recall that $\varepsilon\Delta(t) \equiv \|U^1_\varepsilon(t) - U^\varepsilon(t)\|_s \leqslant C\,\|W(t)\|_s$. Then $\Delta^2(t) \leqslant C\phi(t)$ due to (6.22) and $\phi(0) = C(T_1+1)\,\varepsilon$. Moreover,

$$\phi'(t) \leqslant C(1+B_\varepsilon(t))\,\phi(t) + C(1+B_\varepsilon(t))\,\phi^{(s+3)/2}(t).$$

Recall that $\sup_\varepsilon \int_0^{T_1} B_\varepsilon(t)\,dt < \infty$. By Lemma 6.3, we have

$$\Delta^2(t) \leqslant C\phi(t) \leqslant C\exp\left(C\int_0^{T_1}(1+B_\varepsilon(t))\,dt\right)$$

if ε is so small that $\phi(0)=C(T_1+1)\,\varepsilon \leqslant \delta$. Thus, $\Delta(t)$ is uniformly bounded and it follows from (6.22) that

$$\sup_{t\in[0,\,T]} \|W(t)\|_s^2 \leqslant C(T_1+1)\exp(C(T_1+1))\,\varepsilon^3.$$

This completes the proof.

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